2009 REU Program at the University of West Georgia

Sample Research Problems

Note: These may not coincide with actual problems that we will be encounter this summer, but they are certainly representative of the types of problems that we will be working on.

A. Mixed van der Waerden Numbers. The classical van der Waerden number, $w(k;r)$, is defined as the least positive integer n such that for every partition of $\{1, 2, ..., n\}$ into r subsets, there is some subset containing a k-term arithmetic progression (a.p.). Using the language of "colorings," $w(k;r)$ is defined as the least positive integer n such that every r-coloring of $\{1, 2, ..., n\}$ admits a monochromatic k-term a.p. Slightly generalizing this concept, the (mixed) van der Waerden number $w(k_0, k_1, ..., k_{r-1}; r)$ is defined as the least positive integer n such that for every r-coloring $\chi : 1, 2, ..., n \to \{0, 1, ..., r-1\}$ there is, for some i, $0 \leq i \leq r-1$, a k_i-term a.p. of color i. Although the (classical) van der Waerden numbers $w(k;r)$ have received much attention for almost eighty years (with most of fundamental questions still unanswered), relatively little attention has been given to the mixed (or "off-diagonal") van der Waerden numbers when compared to, say, the classical (mixed) graph-theoretical Ramsey numbers $R(k_1, k_2, ..., k_r)$. In recent work by Culver, Landman, and Robertson, and then by Khodkar and Landman, several new exact values of the mixed van der Waerden numbers have been found. In particular, for the special case of $w(k, 2, 2, ..., 2; r)$, exact formulas are given in terms of k and r. There are several intriguing, but probably not extremely difficult, questions remaining. One such question, which seems a natural "next step," is to find reasonably good bound(s) (upper or lower) on $w(3, 3, 2, 2, ..., 2; r)$. A bit of a generalization of this problem would be to get some information on the magnitude of $w(3, 3, ..., 3, 2, 2, ..., 2; r)$; put another way, we want to study the magnitude of $n(\ell, m) = w(3, 3, ..., 3, 2, 2, ..., 2; r)$, where there are ℓ 3's and m 2's. In addition, almost no work has been done on $w(k, 2, 2, ..., 2; r)$ for the cases in which $k < r$.

B. Permutations of $\mathbf{Z}^+$ without Monotone A.P.'s. A sequence of integers $(a_1, a_2, \ldots)$ has a *monotone k-term a.p.* if there is a set of indices $\{i_1 < i_2 < \ldots < i_k\}$ such that the subsequence $a_{i_1}, a_{i_2}, \ldots, a_{i_k}$ is either an increasing or a decreasing a.p. Davis, Entringer, Graham, and Simmons proved that every permutation of $\mathbf{Z}^+$ must have a 3-term monotone a.p. On the other hand, they showed that there exist permutations of $\mathbf{Z}^+$ that do not have any 5-term monotone a.p.'s. An intriguing questions is whether there are any permutations of $\mathbf{Z}^+$ that avoid 4-term monotone a.p.'s. Another problem that remains wide open for investigation is the question of avoiding monotone a.p.'s when $\mathbf{Z}$ is permuted.

C. Schur Numbers. The Schur number $s(r)$, $r \geq 1$, is the least positive integer such that for every r-coloring of $\{1, 2, \ldots, s(r)\}$ there is a monochromatic solution to $x + y = z$. The only values known for the Schur numbers are $s(1) = 2$, $s(2) = 5$, $s(3) = 14$, and $s(4) = 45$. An open conjecture is that $s(5) = 160$. There are also many refinements and offshoots of the Schur function $s(r)$ that can be explored.

D. Rainbow Colorings. Ramsey theory has been described as the study of unavoidable regularity within large enough structures. According to T. Motzkin, "complete disorder is impossible." Only in the last few years, the "opposite" idea - (roughly speaking, that "complete disorder is unavoidable" - has seen a flurry of activity. More specifically, we are interested in the existence of r-colorings which produce r-term sequences (for example, a.p.'s) such that no two terms have the same color - appropriately called *rainbow* colorings. Rainbow Ramsey theory in the context of graphs has received much attention, but such is not the case in the context of the positive integers (for example, rainbow counterparts to van der Waerden's theorem, Rado's theorem Schur's theorem). Rainbow Ramsey theory on the integers holds great potential for worthwhile research projects for students. In particular, a rainbow counterpart to Rado's theorem on the partition regularity of a system of linear equations, which seems quite plausible, has been conjectured by Jungic, Nesetril, and Radiocic.

E. Spectrum of critical sets in Latin squares. A *partial Latin square* P of order n is an $n \times n$ array with rows and columns indexed by $N = \{0, 1, 2, 3, \ldots, n-1\}$ and entries chosen from N in such a way that each element of N occurs at most once in each row and at most once in each column of the array.

0	1		
1			
			2

A: size 4

			0
	3	0	
3			2

B: size 5

0	1	2	
1	2		
2			

C: size 6

0		2	
	2		0
2		0	
3		1	2

D: size 9

0	1	2	3
1	2	3	0
2	3	0	1
3	0	1	2

E: size 16

Figure 1: Four partial Latin squares and a Latin square of order 4

If all the cells of the array are filled then the partial Latin square is termed a *Latin square* (see Figure 1).
A partial Latin square $\mathcal{C}$ of order n is a *critical set* if $\mathcal{C}$ is contained in exactly one Latin square of order n and no proper subset of $\mathcal{C}$ satisfies this property. Partial Latin squares A, B and C in Figure 1 are critical sets in Latin square E. It is easy to see that there are two Latin squares of order four containing partial Latin square D. Hence, D is not a critical set.
The *spectrum* of critical sets in a Latin square L is a set S of integers such that for every $m \in S$ there is a critical set of size m in L. It is well-known that the spectrum of critical sets in Latin square E in Figure [1] is $S = \{4, 5, 6\}$.
The general problem of determining the spectrum of a Latin square of order n is quite difficult, but for many significant classes of Latin squares it is manageable.
A *back circulant* Latin square of order n is a Latin square whose cell (i, j) contains $i + j \pmod n$ for all $i, j \in \mathbf{Z}_n$. Recently, it was proved that if n is odd, then there exists a critical set of size m in the back circulant Latin square of order n for every $m \in \{n^2/4, (n^2/4) + 1, (n^2/4) + 2, \ldots, (n^2 - n)/2\}$. The problem for n even is still open.

F. Domination in graphs: A graph consists of a set of *vertices (points)* and a set of *edges (lines)*. Each edge of a graph joins two vertices of that graph. Figure 2 shows the vertices and the edges of the complete grid graph $P_4 \times P_7$. A vertex *dominates* itself and all the vertices which are joined to this vertex. For example vertex u_1^1 dominates vertices u_1^1, u_1^2, u_2^1 and vertex u_3^4 dominates vertices $u_3^3, u_3^4, u_3^5, u_2^4, u_4^4$. A subset S of vertices is called a *dominating set* if every vertex of graph is either an element of S or is dominated by an element of S. For example

$$S_1 = \{u_1^1, u_1^4, u_1^7, u_2^3, u_2^5, u_3^2, u_3^4, u_3^6, u_4^1, u_4^3, u_4^5, u_4^7\}$$

is a dominating set (of size 12) for $P_4 \times P_7$. A graph can have many different dominating sets. A dominating set of smaller size is of more interest. A *minimum* dominating set for a graph is a dominating set with smallest number of vertices. For example

$$S_2 = \{u_1^2, u_1^6, u_2^4, u_2^7, u_3^1, u_3^5, u_4^3, u_4^7\}$$

is a minimum dominating set for $P_4 \times P_7$.

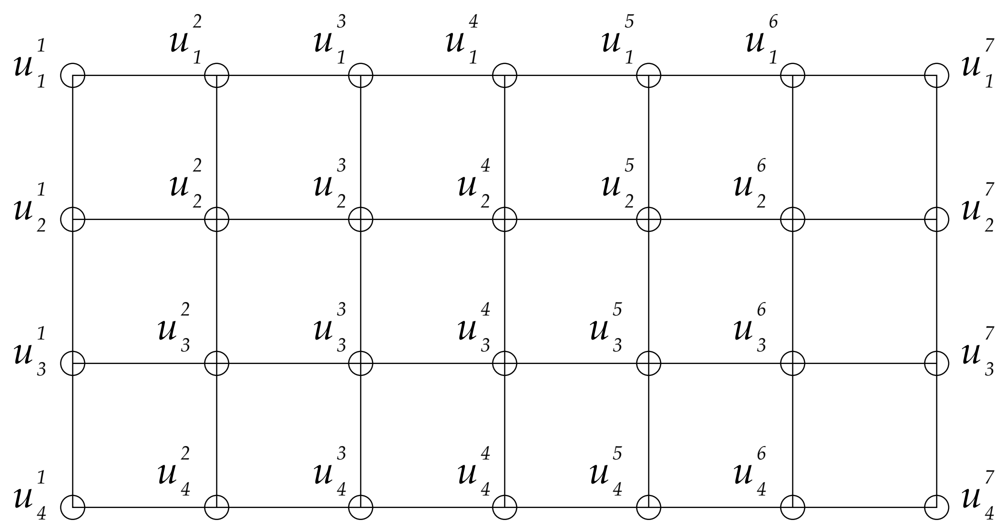

Figure 2: $P_4 \times P_7$

"Hot" topics concerning domination in graphs include: fundamentals of domination in graphs; clique/connected/total domination perfect graphs; domination in complete grid graphs; domination in Harary graphs; double domination numbers; perfect dominating sets; and forcing domination numbers.

G. Jumping Champions. As coined by J.H. Conway, a *jumping champion*, $C(x)$, for a positive integer x, is the most frequently occurring difference $p_i - p_{i-1}$ between consecutive primes that do not exceed x. For example, the jumping champion for $x = 12$ is 2 since the differences between consecutive primes not exceeding 12 are: 1, 2, 2, and 4. In Richard Guy's book of unsolved problems, several questions. originally from a paper by Odlyzko, Rubinstein, and Wolf, on jumping champions are posed. Are there any champions besides 4 and the factorials of primes (2!, 3!, 5!, ...) (the evidence so far says no)? Is it true that $\lim_{x\to\infty} C(x) = \infty$?