Consider the operator equation in $X$

$$L_1X - XL_2 = Y$$  \hspace{1cm} (1)

where $Y$, $L_1$ and $L_2$ are given operators. When $L_1$ and $L_2$ are bounded operators, one can prove the existence and uniqueness of a solution $X$,

$$X = \frac{1}{2\pi i} \int_{\Gamma} \left( (L_1 - \lambda I)^{-1} Y (L_2 - \lambda I)^{-1} \right) d\lambda$$

and (1) has a unique solution if and only if $L_1X = XL_2$ has the trivial solution only. In the simple case when $L_1$ and $L_2$ are finite matrices with disjoint spectra, then $L_1X = XL_2$ has the trivial solution $X = 0$, which is contained in the Sylvester-Roseblum theorem. In this talk we show why uniqueness may not hold, when $L_1$ and $L_2$ are unbounded operators. The main idea is to use transmutation operators between $L_1$ and $L_2$ to construct a non trivial solution.

All are welcome.