ON THE PALEY-WIENER THEOREM

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Dedicated to the memory of Academician M.M. Djrbashian

Abstract. The Fourier transforms of functions with compact and convex supports as well as with non-convex and unbounded supports are considered.

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In this paper we describe real-variable characteristics of Fourier transforms of functions with compact supports in R^n. Real-variable descriptions of Fourier transforms of functions with nonconvex or unbounded supports are also considered.

Theorem 1. *Let* $f \in C^\infty(R^n)$ *such that all Laplacians* $\Delta^k f, \quad k = 0, 1, \ldots,$ *belong to* $L_2(R^n)$. *Then there always exists the limit*

$$\sigma_f = \lim_{k\to\infty} \|\Delta^k f\|_2^{1/(2k)}, \tag{1}$$

and moreover,

$$\sigma_f = \sup\{|\xi| : \quad \xi \in \mathrm{supp}\hat{f}(\xi)\}, \tag{2}$$

where $\hat{f}(\xi)$ *is the Fourier transform of the function* $f(x)$ *(see [2]):*

$$\hat{f}(\xi) = (2\pi)^{-n/2} \int_{R^n} f(x) e^{ix.\xi} dx, \tag{3}$$

and the support of a function f *is the smallest closed set such that* $f = 0$ *almost everywhere in its complement.*

The proof is based on the Plancherel theorem for the Fourier transform and the inequalities

$$(\sigma_f - \epsilon)^{4k} \int_{\sigma_f > |\xi| > \sigma_f - \epsilon} |\hat{f}(\xi)|^2 d\xi \leq \|\Delta^k f\|_2^2 = \int_{|\xi| < \sigma_f} |\xi|^{4k} |\hat{f}(\xi)|^2 d\xi \leq \sigma_f^{4k} \|f\|_2^2. \tag{4}$$

Corollary. *A square integrable in* R^n *function* f *is the restriction of an entire function of exponential type* σ_f *if and only if* $\Delta^k f$ *belong to* $L_2(R^n)$ *for all positive integers* k *and formula (1) holds.*

The proof follows from the Paley-Wiener theorem [2, 3] and Theorem 1.

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Theorem 2. *Let* $f \in L_2(R^n)$ *and*

$$g_k(x) = (4\pi k)^{-n/2} \int_{R^n} f(y) \exp(-|x-y|^2/(4k))dy, \quad k = 0, 1, \ldots. \tag{5}$$

Then there always exists the limit

$$\exp(-\delta_f^2) \doteq \lim_{k\to\infty} \|g_k\|_2^{1/(k)}, \tag{6}$$

and moreover,

$$\delta_f = \inf\{|\xi| : \quad \xi \in \mathrm{supp}\hat{f}(\xi)\}. \tag{7}$$

The proof follows from the inequalities

$$\begin{aligned} \exp(-2k(\delta_f+2\epsilon)^2) \int_{|\xi-\xi_0|<\epsilon} |\hat{f}(\xi)|^2 d\xi \le \|g_k\|_2^2 &= \int_{|\xi|>\delta_f} \exp(-2k|\xi|^2)|\hat{f}(\xi)|^2 d\xi \\ &\le \exp(-2k\delta_f^2)\|f\|_2^2. \end{aligned} \tag{8}$$

Remark. *The following inequality is always true*

$$\lim_{k\to\infty} \|\Delta^k f\|_2^{1/(k)} \ge -\lim_{k\to\infty} \frac{1}{k} \ln \|g_k\|_2, \tag{9}$$

and a square integrable in R^n *function* f *is the Fourier transform of a function with the support lying in the domain* $\delta_f \le |\xi| \le \sigma_f$ *if and only if*

$$\sigma_f^2 \ge \lim_{k\to\infty} \|\Delta^k f\|_2^{1/(k)} \ge -\lim_{k\to\infty} \frac{1}{k} \ln \|g_k\|_2 \ge \delta_f^2. \tag{10}$$

According to [3], a convex, compact and symmetric subset in R^n with nonempty interior is called a symmetric body. Let K be a symmetric body in R^n. The set $K^* = \{y \in R^n; \quad x.y \le 1 \quad \text{for all} \quad x \in K\}$ is called the polar set of K. It is known that K^* is also a symmetric body and $(K^*)^* = K$ [3].

Theorem 3. *Let* $f(x) \in C^\infty(R^n)$. *Then* f *is the Fourier transform of a square integrable function vanishing outside a symmetric body* K *if and only if partial derivatives* $D^\alpha f$ *belong to* $L_2(R^n)$ *for all multi-indices* $\alpha = (\alpha_1, \ldots, \alpha_n)$ *and*

$$\sup_{a\in K^*} \|(a.D)^k f(x)\|_2 < \infty, \quad k = 1, 2, \ldots, \tag{11}$$

where $D = (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})$.

The proof follows from the Plancherel theorem

$$\|(a.D)^k f(x)\|_2 = \|(a.\xi)^k \hat{f}(\xi)\|_2. \tag{12}$$

and the property that $|a.\xi| \le 1$ for all $\xi \in K$ and $a \in K^*$.

Let $P(x)$ be a non-constant polynomial and $\Omega_P = \{x \in R^n : \quad |P(x)| \le 1\}$. Ω_P is called a polynomial domain in R^n. A ball is a polynomial domain. A polynomial domain (for example, $U = \{x : \quad |x_1 x_2 \dots x_n| \leq 1\}$) may be unbounded and nonconvex. Let $\mathcal{S}(R^n)$ be the space of infinitely differentiable functions approaching 0 at infinity together with all its derivatives faster than any polynomial of $|x|^{-1}$. Then we have

Theorem 4. *A function $f \in \mathcal{S}(R^n)$ is the Fourier transform of a function vanishing outside a polynomial domain Ω_P if and only if*

$$\overline{\lim_{k\to\infty}} \|P^k(iD)f(x)\|_p^{1/k} \leq 1, \quad 1 < p < \infty. \tag{13}$$

Proof. First we notice that the theorem has to be proved only for $f \not\equiv 0$. We see that the Fourier transform of $P^k(iD)f(x)$ is $P^k(\xi)\hat{f}(\xi)$.

i) Let now $1 < p < 2$. Suppose that (13) is valid. Applying the Hausdorff-Young inequality ([3])

$$(2\pi)^{n(1/q-1/2)} \|P^k(iD)f(x)\|_p \geq \|P^k(\xi)\hat{f}(\xi)\|_q, \quad p^{-1} + q^{-1} = 1, \tag{14}$$

we have

$$\overline{\lim_{k\to\infty}} \|P^k(\xi)\hat{f}(\xi)\|_q^{1/k} \leq 1 \quad . \tag{15}$$

Let $\xi_0 \notin \Omega_P$, that means $|P(\xi_0)| > 1$. Let U_{ξ_0} be a neighborhood of ξ_0 with the property $|P(\xi)| > (1 + |P(\xi_0)|)/2 > 1$ for $\xi \in U_{\xi_0}$. Then

$$\begin{aligned} 1 &\geq \overline{\lim_{k\to\infty}} \|P^k(\xi)\hat{f}(\xi)\|_q^{1/k} \geq \overline{\lim_{k\to\infty}} \left\{ \int_{U_{\xi_0}} |P^k(\xi)\hat{f}(\xi)|^q d\xi \right\}^{1/(qk)} \\ &\geq \frac{(1 + |P(\xi_0)|)}{2} \overline{\lim_{k\to\infty}} \left\{ \int_{U_{\xi_0}} |\hat{f}(\xi)|^q d\xi \right\}^{1/(qk)} . \end{aligned} \tag{16}$$

Because the last limit in (16) can be either 1 or 0, then

$$\overline{\lim_{k\to\infty}} \left\{ \int_{U_{\xi_0}} |\hat{f}(\xi)|^q d\xi \right\}^{1/(qk)} = 0,$$

that means $\int_{U_{\xi_0}} |\hat{f}(\xi)|^q d\xi = 0$. Consequently, $\xi_0 \notin \text{supp}\hat{f}$, and hence, $\text{supp}\hat{f} \subseteq \Omega_P$.

Let now $\text{supp}\hat{f} \subseteq \Omega_P$. We have

$$\begin{aligned} \|f\|_p^p = \int_{R^n} (1 + |x|^2)^{-np} |(1 + |x|^2)^n f(x)|^p dx &\leq \|(1 + |x|^2)^n f(x)\|_2^p \|(1 + |x|^2)^{-np}\|_{\frac{2}{2-p}} \\ &\leq C\|(1 + |x|^2)^n f(x)\|_2^p = C\|(1 - \Delta)^n \hat{f}(\xi)\|_2^p, \end{aligned} \tag{17}$$

where the Hölder inequality is applied. Therefore,

$$\|P^k(iD)f(x)\|_p \leq C\|(1 - \Delta)^n [P^k(\xi)\hat{f}(\xi)]\|_2. \tag{18}$$

By induction one can show that

$$(1 - \Delta)^n [P^k(\xi)\hat{f}(\xi)] = k^{2n} P^{k-2n}(\xi)\phi_k(\xi), \quad k > 2n, \tag{19}$$

with supp $\phi_k \subseteq \mathrm{supp}\hat{f}, \quad \|\phi_k\|_2 \le C_1$, where C_1 does not depend on k. Hence,

$$\|P^k(iD)f(x)\|_p \le Ck^{2n}\|[P^{k-2n}(\xi)\phi_k(\xi)]\|_2 \le Ck^{2n}\|\phi_k(\xi)]\|_2 \le CC_1k^{2n}. \tag{20}$$

since $|P^{k-2n}(\xi)| \le 1$ on the support of $\phi_k(\xi)$. Hence the inequality (13) is followed.

ii) Let now $p \ge 2$. Suppose that $\mathrm{supp}\hat{f} \subseteq \Omega_P$. Then $|P(\xi)| \le 1$ on the support of $\hat{f}$ and we have by the Hausdorff-Young inequality

$$(2\pi)^{n(1/2-1/p)}\|P^k(iD)f(x)\|_p \le \|P^k(\xi)\hat{f}(\xi)\|_q \le \|\hat{f}(\xi)\|_q. \tag{21}$$

Consequently,

$$\overline{\lim_{k\to\infty}}\|P^k(iD)f(x)\|_p^{1/k} \le \overline{\lim_{k\to\infty}}(2\pi)^{\frac{n}{k}(1/p-1/2)}\|\hat{f}(\xi)\|_q^{1/k} = 1. \tag{22}$$

Suppose now that (13) is valid. Applying the Pitt inequality for the n-dimensional Fourier transform

$$\||\xi|^{-\beta}\hat{f}(\xi)\|_r \le \||x|^{\alpha}f(x)\|_p, \quad 0 \le \beta < n, 0 \le \alpha = \beta + n(1-1/p-1/r), 1 < p \le r < \infty,$$

with $r = p > 2, \beta = 0, \alpha = n(1-2/p)$ and notice that $|x|^{n(1-2/p)} < (1+|x|^2)^n$, we have

$$\|P^k(\xi)\hat{f}(\xi)\|_p \le C\|(1+|x|^2)^nP^k(iD)f(x)\|_p. \tag{23}$$

The Fourier transform of $(1+|x|^2)^nP^k(iD)f(x)$ is $(1-\Delta)^nP^k(\xi)\hat{f}(\xi)$. By induction one can show that

$$(1-\Delta)^n[P^k(\xi)\hat{f}(\xi)] = P^{k-2n}(\xi)\sum_{j=1}^{m}a_j(k)\phi_j(\xi), \quad k > 2n, \tag{24}$$

where $a_j(k)$ are polynomials of k with degrees not extending $2n$, ϕ_j are functions from $\mathcal{S}(R^n)$, and m, ϕ_j and coefficients of polynomials a_j do not depend on k. Consequently,

$$(1+|x|^2)^nP^k(iD)f(x) = P^{k-2n}(iD)\sum_{j=1}^{m}a_j(k)\hat{\phi}_j(x). \tag{25}$$

Hence,

$$\|P^k(\xi)\hat{f}(\xi)\|_p \le C\sum_{j=1}^{m}a_j(k)\|P^{k-2n}(iD)\hat{\phi}_j(x)\|_p. \tag{26}$$

We have

$$\left\{\sum_{j=1}^{m}|a_j(k)|\|P^{k-2n}(iD)\hat{\phi}_j(x)\|_p\right\}^{1/k} \le m^{1/k}\sup_{1\le j\le m}a_j^{1/k}(k)\|P^{k-2n}(iD)\hat{\phi}_j(x)\|_p^{1/k}, \tag{27}$$

and because $\hat{\phi}_j \in \mathcal{S}(R^n)$ then

$$\overline{\lim_{k\to\infty}}\|P^{k-2n}(iD)\hat{\phi}_j(x)\|_p^{1/k} \le 1, \quad 1 \le j \le m.$$

The degree of $a_j(k)$ is not extending $2n$, therefore,

$$\overline{\lim_{k\to\infty}}|a_j(k)|^{1/k} \leq 1.$$

Consequently,

$$\overline{\lim_{k\to\infty}}\|P^k(\xi)\hat{f}(\xi)\|_p^{1/k} \leq 1. \tag{28}$$

From here as above we can conclude that $\mathrm{supp}\hat{f}(\xi) \subseteq \Omega_P$, and Theorem 4 is thus proved.

References

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