SOME APPLICATIONS OF THE CONVOLUTION THEOREM OF THE HILBERT TRANSFORM

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Abstract

For the Hilbert transform

$$\widetilde{f}(x) = \frac{1}{\pi}\int_R \frac{f(t)}{x-t}dt$$

a new proof of the convolution formula is given. This convolution formula is then applied to calculate some Cauchy integrals and to solve a nonlinear singular integral equation

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1. INTRODUCTION

Applications of the convolution formulae of Fourier, Laplace and Mellin transforms are well-known. Recently some applications of the convolutions formulae for Hankel, Stieltjes and Kontorovich–Lebedev transforms are given (see [4, 5, 6, 7, 9]). For the Hilbert transform ([1])

$$H[f](x) = \widetilde{f}(x) = \frac{1}{\pi}\int_R \frac{f(t)}{x-t}dt, \tag{1}$$

the convolution theorem has been established in L_p spaces in ([8], p. 169), but is missing in modern textbooks on integral transforms. In this paper we give an another proof of this

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theorem and then apply this result to calculate some Cauchy integrals of special functions and to obtain explicit solutions of a nonlinear singular integral equation.

2. CONVOLUTION THEOREM

Let $f,\ g$ be defined on R and belong corresponding to $L_p(R)$ and $L_q(R),\ 1 < p, q < \infty,\ \ p^{-1} + q^{-1} < 1$. Then Hilbert transforms $\widetilde{f}$ and $\widetilde{g}$ of f and g exist and belong to $L_p(R)$ and $L_q(R)$, too. Furthermore, $fg \in L_r(R)$ with $r^{-1} = p^{-1} + q^{-1}$. Consequently, the Hilbert transform $\widetilde{fg}$ of fg exists and belongs to $L_r(R)$. Therefore, if we put

$$h(x) = (f \otimes g)(x) = \frac{1}{\pi} \int_R (f(x)g(t) + g(x)f(t) - f(t)g(t)) \frac{dt}{x - t}, \tag{2}$$

then h exists and belongs to $L_r(R)$. Our main result in this paragraph is a new proof of the following

Theorem ([8], p. 169). The Hilbert transform of h is the product of the Hilbert transforms of f and g

$$\widetilde{h}(x) = \widetilde{f}(x)\widetilde{g}(x). \tag{3}$$

Proof. Let f and g belong to S, the space of infinitely differentiable functions which, together with their derivatives, approach zero more rapidly than any power of $|x|^{-1}$ as $|x| \to \infty$. Applying the Hilbert transform to the function $h(x)$ we obtain

$$\widetilde{h} = \widetilde{f\widetilde{g}} + \widetilde{g\widetilde{f}} + fg. \tag{4}$$

Applying now the Fourier transform

$$F[f](x) = \int_R f(t) \exp(-ixt)\, dt, \tag{5}$$

to (4) and using the properties [1]

$$F[\widetilde{f}] = -i \operatorname{sgn} x F[f] \tag{6}$$

and $2\pi F[fg] = F[f] \odot F[g]$, where

$$f \odot g = \int_R f(t)g(x - t)\, dt \tag{7}$$

is the Fourier convolution, we get

$$\begin{aligned} 2\pi F[\widetilde{h}] &= 2\pi F[\widetilde{f\widetilde{g}} + \widetilde{\widetilde{f}g} + fg] \\ &= -2\pi i \operatorname{sgn} x F[f\widetilde{g} + \widetilde{f}g] + F[fg] \\ &= -i \operatorname{sgn} x \{F[f] \odot F[\widetilde{g}] + F[\widetilde{f}] \odot F[g]\} + F[f] \odot F[g] \\ &= -\operatorname{sgn} x \{F[f] \odot (\operatorname{sgn} x F[g]) + (\operatorname{sgn} x F[f]) \odot F[g]\} + F[f] \odot F[g] \\ &= (-i \operatorname{sgn} x F[f]) \odot (-i \operatorname{sgn} x F[g]) = (F[\widetilde{f}]) \odot (F[\widetilde{g}]). \end{aligned}$$

Consequently,

$$\widetilde{h} = \widetilde{f}\widetilde{g},$$

that means h is the convolution of the Hilbert transform.
Since the space S is dense in $L_p(R)$ and $L_q(R)$, where the Hilbert transform is bounded, formula (3), first proved to be valid on dense subspaces of $L_p(R)$ and $L_q(R)$, still holds for all $f \in L_p(R)$ and $g \in L_q(R)$. Thus Theorem is proved.

3. EVALUATION OF SOME CAUCHY INTEGRALS

Let $g = \widetilde{f}$. Then formula (2) becomes

$$h = -f^2 + \widetilde{f}^2 - \widetilde{f\widetilde{f}}. \tag{8}$$

But $\widetilde{h} = \widetilde{f}\widetilde{g} = -f\widetilde{f}$. Therefore, $h = \widetilde{f\widetilde{f}}$. Consequently, we have

$$\widetilde{f}^2(x) - f^2(x) = \frac{2}{\pi}\int_R \frac{f(t)\widetilde{f}(t)}{x-t}dt. \tag{9}$$

Formula (9) can be applied to evaluate new Hilbert integrals. Namely, if the Hilbert transform of f is known, then the Hilbert transform of $f\widetilde{f}$ is $\frac{1}{2}\left(\widetilde{f}^2 - f^2\right)$. For example, let $f(x) = \exp(-|x|)I_0(x) \in L_p(R)$. Then $\widetilde{f}(x) = 2\sinh(x)K_0(x)$ (see [2], p. 260). Therefore,

$$\int_R \frac{\exp(-|t|)\sinh(t)K_0(t)I_0(t)}{x-t}dx = \sinh^2(x)K_0^2(x) - \frac{1}{4}\exp(-2|x|)I_0^2(x). \tag{10}$$

Let

$$\begin{aligned} f(x) = G^{m\,n}_{p\,q}\left(ax^2 \middle| {a_1,\dots,a_p \atop b_1,\dots,b_q}\right), \quad & p+q < 2(m+n), |\arg a| < (m+n-p/2-q/2)\pi, \\ & \mathcal{R}a_j < 1, \quad j = 1,\dots,n; \quad \mathcal{R}b_j > -1/2, \quad j = 1,\dots,m, \end{aligned} \tag{11}$$

where $G^{m\,n}_{p\,q}\left(x \middle| {a_1,\dots,a_p \atop b_1,\dots,b_q}\right)$ is the G-Meyer function ([2]). Then ([2], p. 262)

$$\widetilde{f}(x) = -\operatorname{sgn} x G^{m+1\,n+1}_{p+2\,q+2}\left(ax^2 \middle| {1/2,a_1,\dots,a_p,1 \atop 1/2,b_1,\dots,b_q,1}\right). \tag{12}$$

Hence

$$\begin{gathered} \frac{2}{\pi}\int_R \operatorname{sgn} t G^{m\,n}_{p\,q}\left(at^2 \middle| {a_1,\dots,a_p \atop b_1,\dots,b_q}\right) G^{m+1\,n+1}_{p+2\,q+2}\left(at^2 \middle| {1/2,a_1,\dots,a_p,1 \atop 1/2,b_1,\dots,b_q,1}\right)\frac{dt}{x-t} \\ = \left\{G^{m\,n}_{p\,q}\left(ax^2 \middle| {a_1,\dots,a_p \atop b_1,\dots,b_q}\right)\right\}^2 - \left\{G^{m+1\,n+1}_{p+2\,q+2}\left(ax^2 \middle| {1/2,a_1,\dots,a_p,1 \atop 1/2,b_1,\dots,b_q,1}\right)\right\}^2, \\ p+q < 2(m+n), \quad |\arg a| < (m+n-p/2-q/2)\pi, \\ \mathcal{R}a_j < 1, \quad j = 1,\dots,n; \quad \mathcal{R}b_j > -1/2, \quad j = 1,\dots,m, \end{gathered} \tag{13}$$

Using tables of the Hilbert transform one can calculate new Cauchy integrals by this method.

4. A NONLINEAR SINGULAR INTEGRAL EQUATION

Consider now a nonlinear singular integral equation

$$\lambda f(x) + \frac{2}{\pi} f(x) \int_R \frac{f(t)}{x-t} dt - \frac{1}{\pi} \int_R \frac{f^2(t)}{x-t} dt = g(x), \quad \lambda \in C. \tag{14}$$

This equation can be rewritten in the equivalent form

$$\lambda f(x) + (f \otimes f)(x) = g(x). \tag{15}$$

Applying now the Hilbert transform to (15) and using Theorem we have

$$\lambda \widetilde{f} + \widetilde{f}^2 = \widetilde{g}. \tag{16}$$

Solving this equation we obtain

$$\widetilde{f}(x) = -\frac{\lambda}{2} \pm \sqrt{\frac{\lambda^2}{4} + \widetilde{g}(x)}. \tag{17}$$

Here $\sqrt{\frac{\lambda^2}{4} + \widetilde{g}(x)}$ is a branch of the square such that $\Re\left\{\sqrt{\frac{\lambda^2}{4} + \widetilde{g}(x)}\right\} \geq 0$. Let $\lambda = 0$. If $f \in L_p(R)$, then $g \in L_{p/2}(R)$ and therefore, $\widetilde{g} \in L_{p/2}(R)$. We have

$$\widetilde{f}(x) = \pm\sqrt{\widetilde{g}(x)}. \tag{18}$$

Taking

$$\widetilde{f}_\Omega(x) = \begin{cases} \sqrt{\widetilde{g}(x)} & \text{if } x \in \Omega \\ -\sqrt{\widetilde{g}(x)} & \text{otherwise} \end{cases}, \tag{19}$$

where Ω is any measurable subset of R. It is not difficult to see that f_Ω consist all of solutions of the equation (14).
Let $\lambda \neq 0$. We choose

$$\widetilde{f}_\Omega(x) = \begin{cases} -\frac{\lambda}{2} + \sqrt{\frac{\lambda^2}{4} + \widetilde{g}(x)} & \text{if } x \in \Omega \\ -\frac{\lambda}{2} - \sqrt{\frac{\lambda^2}{4} + \widetilde{g}(x)} & \text{otherwise} \end{cases}, \tag{20}$$

It is easy to see that if f is a solution of (14), then its Hilbert transform has the form (20). But not every $\widetilde{f}_\Omega$ belongs to $L_p(R)$. We show that $\widetilde{f}_\Omega \in L_p(R)$ if and only if

$$\begin{array}{ll} |\Omega| < \infty & \text{if } \Re\lambda < 0 \\ |R/\Omega| < \infty & \text{otherwise} \end{array}, \tag{21}$$

where $|\Omega|$ is the measure of Ω. Indeed, let $\Re\lambda < 0$. Then

$$\|\widetilde{f}_\Omega\|_p^p \geq \int_\Omega \left| -\frac{\lambda}{2} + \sqrt{\frac{\lambda^2}{4} + \widetilde{g}(x)} \right|^p dx \geq \left|\frac{\lambda}{2}\right|^p |\Omega|.$$

Therefore, if $\widetilde{f}_\Omega \in L_p(R)$, then $|\Omega| < \infty$. We prove that this condition is not only neccesary, but also sufficient. We have

$$\|\widetilde{f}_\Omega\|_p^p = \|\widetilde{f}_\Omega\|_{L_p(\Omega)}^p + \left|\frac{2}{\lambda}\right|^p \int_{R/\Omega} \left|\frac{\widetilde{g}(x)}{1+\sqrt{1+4\lambda^{-2}\widetilde{g}(x)}}\right|^p dx$$
$$\leq \|\widetilde{f}_\Omega\|_{L_p(\Omega)}^p + \left|\frac{2}{\lambda}\right|^p \|\widetilde{g}\|_p < \infty.$$

Analogously for the case $\Re\lambda \geq 0$.
Therefore, all solutions of the equation (14) are Hilbert transforms of $-\widetilde{f}_\Omega$ having form (20) with the condition (21).

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