SOME APPLICATIONS OF THE CONVOLUTION THEOREM OF THE HILBERT TRANSFORM

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Abstract

For the Hilbert transform
\[ \tilde{f}(x) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{f(t)}{x-t} dt \]
a new proof of the convolution formula is given. This convolution formula is then applied to calculate some Cauchy integrals and to solve a nonlinear singular integral equation.

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1. INTRODUCTION

Applications of the convolution formulae of Fourier, Laplace and Mellin transforms are well-known. Recently some applications of the convolutions formulae for Hankel, Stieltjes and Kontorovich–Lebedev transforms are given (see [4, 5, 6, 7, 9]). For the Hilbert transform ([1])
\[ H[f](x) = \tilde{f}(x) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{f(t)}{x-t} dt, \tag{1} \]
the convolution theorem has been established in $L_p$ spaces in ([8], p. 169), but is missing in modern textbooks on integral transforms. In this paper we give an another proof of this

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theorem and then apply this result to calculate some Cauchy integrals of special functions and to obtain explicit solutions of a nonlinear singular integral equation.

2. CONVOLUTION THEOREM

Let \( f, g \) be defined on \( R \) and belong corresponding to \( L_p(R) \) and \( L_q(R) \), \( 1 < p, q < \infty, \ p^{-1} + q^{-1} < 1 \). Then Hilbert transforms \( \tilde{f} \) and \( \tilde{g} \) of \( f \) and \( g \) exist and belong to \( L_p(R) \) and \( L_q(R) \), too. Furthermore, \( fg \in L_r(R) \) with \( r^{-1} = p^{-1} + q^{-1} \). Consequently, the Hilbert transform \( \tilde{fg} \) of \( fg \) exists and belongs to \( L_r(R) \). Therefore, if we put \( h(x) = (f \otimes g)(x) = \frac{1}{\pi} \int_R (f(x)g(t) + g(x)f(t) - f(t)g(t)) \frac{dt}{x-t} \), (2)

then \( h \) exists and belongs to \( L_r(R) \). Our main result in this paragraph is a new proof of the following

**Theorem** ([8], p. 169). The Hilbert transform of \( h \) is the product of the Hilbert transforms of \( f \) and \( g \)

\[ \tilde{h}(x) = \tilde{f}(x)\tilde{g}(x). \] (3)

**Proof.** Let \( f \) and \( g \) belong to \( S \), the space of infinitely differentiable functions which, together with their derivatives, approach zero more rapidly than any power of \(|x|^{-1}\) as \(|x| \to \infty\). Applying the Hilbert transform to the function \( h(x) \) we obtain

\[ \tilde{h} = \tilde{f}\tilde{g} + \tilde{g}\tilde{f} + fg. \] (4)

Applying now the Fourier transform

\[ F[f](x) = \int_R f(t) \exp(-ixt) \, dt, \] (5)

to (4) and using the properties [1]

\[ F[\tilde{f}] = -i \text{sgn} \, x F[f] \] (6)

and \( 2\pi F[fg] = F[f] \odot F[g] \), where

\[ f \odot g = \int_R f(t)g(x-t) \, dt \] (7)

is the Fourier convolution, we get

\[
2\pi F[\tilde{h}] = 2\pi F[\tilde{f}\tilde{g} + \tilde{g}\tilde{f} + fg] \\
= -2\pi i \text{sgn} \, x F[f\tilde{g} + \tilde{f}g] + F[fg] \\
= -i \text{sgn} \, x \{ F[f] \odot F[\tilde{g}] + F[\tilde{f}] \odot F[g] \} + F[f] \odot F[g] \\
= -\text{sgn} \, x \{ F[f] \odot (\text{sgn} \, x F[g]) + (\text{sgn} \, x F[f]) \odot F[g] \} + F[f] \odot F[g] \\
= (-i \text{sgn} \, x F[f]) \odot (-i \text{sgn} \, x F[g]) = (F[\tilde{f}]) \odot (F[\tilde{g}]).
\]
Consequently,
\[ \tilde{h} = \tilde{f} \tilde{g}, \]
that means \( h \) is the convolution of the Hilbert transform. Since the space \( S \) is dense in \( L_p(R) \) and \( L_q(R) \), where the Hilbert transform is bounded, formula (3), first proved to be valid on dense subspaces of \( L_p(R) \) and \( L_q(R) \), still holds for all \( f \in L_p(R) \) and \( g \in L_q(R) \). Thus Theorem is proved.

3. EVALUATION OF SOME CAUCHY INTEGRALS

Let \( g = \tilde{f} \). Then formula (2) becomes
\[ h = -f^2 + \tilde{f}^2 - \tilde{f} \tilde{f}. \] (8)
But \( \tilde{h} = \tilde{f} \tilde{g} = -f \tilde{f} \). Therefore, \( h = f \tilde{f} \). Consequently, we have
\[ \tilde{f}^2(x) - f^2(x) = \frac{2}{\pi} \int_R \frac{f(t) \tilde{f}(t)}{x-t} dt. \] (9)

Formula (9) can be applied to evaluate new Hilbert integrals. Namely, if the Hilbert transform of \( f \) is known, then the Hilbert transform of \( f \tilde{f} \) is \( \frac{1}{2} (\tilde{f}^2 - f^2) \). For example, let \( f(x) = \exp(-|x|)I_0(x) \in L_p(R) \). Then \( \tilde{f}(x) = 2 \sinh(x)K_0(x) \) (see [2], p. 260). Therefore,
\[ \int_R \frac{\exp(-|t|) \sinh(t)K_0(t)I_0(t)}{x-t} dx = \sinh^2(x)K_0^2(x) - \frac{1}{4} \exp(-2|x|)I_0^2(x). \] (10)

Let
\[ f(x) = G_{pq}^{mn} \left( ax^2 \left| \begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array} \right. \right), \quad p + q < 2(m + n), \arg a < (m + n - p/2 - q/2)\pi, \quad \Re a_j < 1, \quad j = 1, \ldots, n; \quad \Re b_j > -1/2, \quad j = 1, \ldots, m, \] (11)
where \( G_{pq}^{mn} \left( x \left| \begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array} \right. \right) \) is the G-Meyer function ([2]). Then ([2], p. 262)
\[ \tilde{f}(x) = -\text{sgn} x G_{p+2q+2}^{m+1n+1} \left( ax^2 \left| \begin{array}{c} 1/2, a_1, \ldots, a_p; 1/2, b_1, \ldots, b_q \end{array} \right. \right). \] (12)

Hence
\[ \frac{2}{\pi} \int_R \text{sgn} t G_{pq}^{mn} \left( at^2 \left| \begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array} \right. \right) G_{p+2q+2}^{m+1n+1} \left( at^2 \left| \begin{array}{c} 1/2, a_1, \ldots, a_p; 1/2, b_1, \ldots, b_q \end{array} \right. \right) \frac{dt}{x-t} \]
\[ = \left\{ G_{pq}^{mn} \left( ax^2 \left| \begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array} \right. \right) \right\}^2 - \left\{ G_{p+2q+2}^{m+1n+1} \left( ax^2 \left| \begin{array}{c} 1/2, a_1, \ldots, a_p; 1/2, b_1, \ldots, b_q \end{array} \right. \right) \right\}^2, \] (13)

\[ p + q < 2(m + n), \quad \arg a < (m + n - p/2 - q/2)\pi, \quad \Re a_j < 1, \quad j = 1, \ldots, n; \quad \Re b_j > -1/2, \quad j = 1, \ldots, m, \]

Using tables of the Hilbert transform one can calculate new Cauchy integrals by this method.
4. A NONLINEAR SINGULAR INTEGRAL EQUATION

Consider now a nonlinear singular integral equation

\[ \lambda f(x) + \frac{2}{\pi} f(x) \int_R \frac{f(t)}{x-t} dt - \frac{1}{\pi} \int_R \frac{f^2(t)}{x-t} dt = g(x), \quad \lambda \in C. \quad (14) \]

This equation can be rewritten in the equivalent form

\[ \lambda f(x) + (f \otimes f)(x) = g(x). \quad (15) \]

Applying now the Hilbert transform to (15) and using Theorem we have

\[ \lambda \tilde{f} + \tilde{f}^2 = \tilde{g}. \quad (16) \]

Solving this equation we obtain

\[ \tilde{f}(x) = -\frac{\lambda}{2} \pm \sqrt{\frac{\lambda^2}{4} + \tilde{g}(x)}. \quad (17) \]

Here \( \sqrt{\frac{\lambda^2}{4} + \tilde{g}(x)} \) is a branch of the square such that \( \Re \{ \sqrt{\frac{\lambda^2}{4} + \tilde{g}(x)} \} \geq 0 \). Let \( \lambda = 0 \). If \( f \in L_p(R) \), then \( g \in L_{p/2}(R) \) and therefore, \( \tilde{g} \in L_{p/2}(R) \). We have

\[ \tilde{f}(x) = \pm \sqrt{\tilde{g}(x)}. \quad (18) \]

Taking

\[ \tilde{f}_\Omega(x) = \begin{cases} \sqrt{\tilde{g}(x)} & \text{if } x \in \Omega \\ -\sqrt{\tilde{g}(x)} & \text{otherwise} \end{cases}, \quad (19) \]

where \( \Omega \) is any measurable subset of \( R \). It is not difficult to see that \( f_\Omega \) consist all of solutions of the equation (14).

Let \( \lambda \neq 0 \). We choose

\[ \tilde{f}_\Omega(x) = \begin{cases} -\frac{\lambda}{2} + \sqrt{\frac{\lambda^2}{4} + \tilde{g}(x)} & \text{if } x \in \Omega \\ -\frac{\lambda}{2} - \sqrt{\frac{\lambda^2}{4} + \tilde{g}(x)} & \text{otherwise} \end{cases}, \quad (20) \]

It is easy to see that if \( f \) is a solution of (14), then its Hilbert transform has the form (20). But not every \( \tilde{f}_\Omega \) belongs to \( L_p(R) \). We show that \( \tilde{f}_\Omega \in L_p(R) \) if and only if

\[ |\Omega| < \infty \quad \text{if } \Re \lambda < 0 \]
\[ |R/\Omega| < \infty \quad \text{otherwise}, \quad (21) \]

where \( |\Omega| \) is the measure of \( \Omega \). Indeed, let \( \Re \lambda < 0 \). Then

\[ \|\tilde{f}_\Omega\|_p^p \geq \int_\Omega \left| -\frac{\lambda}{2} + \sqrt{\frac{\lambda^2}{4} + \tilde{g}(x)} \right|^p dx \geq \left| \frac{\lambda}{2} \right|^p |\Omega|. \]
Therefore, if $\tilde{f}_\Omega \in L_p(R)$, then $|\Omega| < \infty$. We prove that this condition is not only necessary, but also sufficient. We have

$$\|\tilde{f}_\Omega\|_p^p = \|\tilde{f}_\Omega\|_{L_p(\Omega)}^p + \left|\frac{2}{\lambda}\int_{R/\Omega} \frac{\tilde{g}(x)}{1 + \sqrt{1 + 4\lambda^{-2}\tilde{g}(x)}} \right|^p \lesssim \|\tilde{f}_\Omega\|_{L_p(\Omega)}^p + \left|\frac{2}{\lambda}\|\tilde{g}\|_p\right|^p < \infty.$$ 

Analogously for the case $\Re \lambda \geq 0$.

Therefore, all solutions of the equation (14) are Hilbert transforms of $-\tilde{f}_\Omega$ having form (20) with the condition (21).

References


