EXPRESSIONS OF LEGENDRE POLYNOMIALS
THROUGH EULER POLYNOMIALS

Vu Kim Tuan
Department of Mathematics and Informatics
Faculty of Science, Kuwait University
P.O. Box 5969, Safat 13060, Kuwait

and

Nguyen Thi Tinh
Hanoi Teacher’s Training College N°1

Abstract
The discrete orthogonality of the modified Lommel polynomials is applied to establish a formula of finite summation that is used to construct a relation between Legendre polynomials and Euler polynomials. Consequently, explicit coefficients of expansions of Legendre polynomials through Euler polynomials are obtained.

Key words: Legendre Polynomials, Euler Polynomials.

1. Introduction
The main aim of this work is to obtain an expansion formula of Legendre polynomials through Euler polynomials.

Legendre polynomials $P_n(x)$ are defined by

$$P_n(x) = \frac{1}{n!2^n} \frac{d^n (x^2 - 1)^n}{dx^n}, \quad n \geq 0,$$

and satisfy the following relation of orthogonality

$$\int_{-1}^{1} P_n(x) P_m(x) dx = \frac{2}{2n + 1} \delta_{mn}, \quad (1)$$

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or equivalently,
\[ \int_0^1 P_n(2x - 1) P_m(2x - 1) \, dx = \frac{1}{2n + 1} \delta_{mn}. \]  
(2)

([3], Chapter 3, 3.12.8 and 3.12.10), where
\[ \delta_{mn} = \begin{cases} 
0, & m \neq n \\
1, & m = n 
\end{cases} \]
is the ”Kronecker delta”.

Euler polynomials \( E_n(x) \) and Bernoulli numbers \( B_n, \quad n \geq 0 \), are defined by
\[ 2 e^{xz} = \sum_{n=0}^{\infty} E_n(x) \frac{z^n}{n!}, \quad |z| < \pi, \]
(3)
\[ z e^z - 1 = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!}, \quad |z| < 2\pi, \]
(4)
(see [3], Chapter 1). Two of the properties of Euler polynomials \( E_n(x) \) and Bernoulli numbers \( B_n \) used later are
\[ B_{2k+1} = 0, \quad k > 0; \]
(5)
\[ \int_0^1 E_m(x) E_n(x) \, dx = 4 (-1)^n \left( \frac{2^{m+n+2} - 1}{(m+n+2)!} \right) \frac{m! n!}{(m+n+2)!} B_{m+n+2} \]
(6)
(see [5], Chapter 2).

Let \( (\nu)_n = \Gamma(\nu+n)/\Gamma(\nu) = \nu (\nu + 1) \ldots (\nu + n - 1) \) be the Pochhammer symbol, \( _2F_3(a_1, a_2; b_1, b_2, b_3; z) \) be a generalized hypergeometric function ([3], Chapter 4). Let \( \{j_{\nu,n}\}, \quad n = \pm 1, \pm 2, \ldots \), be the nonvanishing zeros of the Bessel function \( J_{\nu}(x) \) ([4], Chapter 7) ordered by
\[ \cdots < j_{\nu,-2} < j_{\nu,-1} < 0 < j_{\nu,1} < j_{\nu,2} < \cdots . \]

Then
\[ h_{n,\nu}(x) = (\nu)_n (2x)^n \quad _2F_3 \left( -n/2, (1-n)/2; \nu, -n, 1 - \nu - n; -1/x^2 \right) \quad (7) \]
are modified Lommel polynomials ([2]) satisfying the following discrete orthogonality

\[
\sum_{k=-\infty}^{\infty} h_{n,\nu} \left( \frac{1}{j_{\nu-1,k}} \right) h_{m,\nu} \left( \frac{1}{j_{\nu-1,k}} \right) \frac{1}{j_{\nu}^2} = \frac{\delta_{mn}}{2(\nu+n)},
\]

where the dash in the sum indicates that the term with index \( k = 0 \) is omitted. Formula (8) first appeared in [2] with the incorrect right-hand side.

2. Formula of finite summation

The following formula of finite summation will play an important role in the next section. It was given incorrectly in [5], Chapter 5, 5.1.1.7. For we could not find another reference, we shall establish it here by using the orthogonality relation (8) of the modified Lommel polynomials.

**Lemma.** For \( \sigma = 0 \) or 1 the formula

\[
\sum_{l=0}^{n} \frac{(2n + 2l + 2\sigma)!}{(2l)!(2n-2l)!(2m+2l+2\sigma+2)!} \left( 1 - 2^{2m+2l+2\sigma+2} \right) B_{2m+2l+2\sigma+2} = (-1)^{n+1} (2n + \sigma)! \delta_{mn}, \quad 0 \leq m \leq n,
\]

is valid.

**Proof.** First we consider the case \( \sigma = 0 \). From (7) it is easy to see that the modified Lommel polynomials \( h_{n,\nu}(x) \) is an even or an odd polynomial according as \( n \) is even or odd, i.e. \( h_{n,\nu}(-x) = (-1)^n h_{n,\nu}(x) \). Therefore,

\[
v_n(x) = h_{2n+1/2} \left( \frac{2\sqrt{x}}{\pi} \right), \quad n = 0, 1, \ldots,
\]

is a polynomial of precise degree \( n \) of variable \( x \). From (7) and (10) we have

\[
v_n(x) = \left( \frac{1}{2} \right)_{2n} \frac{4^{2n}x^n}{\pi^{2n}} \sum_{k=0}^{n} \frac{(-n)_k (1/2 - n)_k}{(1)_k (1/2)_k (1/2 - 2n)_k (-2n)_k} \left( \frac{\pi^2}{4x} \right)^k
\]
where \( k!! = k (k - 2) (k - 4) \ldots (k - 2\left\lfloor \frac{k}{2} \right\rfloor + 2) \). Using the formula
\[
4^k \binom{a}{k} (a + 1/2)_k = (2a)_{2k},
\]
we obtain
\[
v_n(x) = \frac{(4n - 1)!! 2^{2n} x^n}{\pi^{2n}} \sum_{k=0}^{n} \frac{(-2n)_{2k}}{(1)_{2k} (4n)_{2k}} \left( -\frac{\pi^2}{x} \right)^k
\]

\[
= \frac{(4n - 1)!! 2^{2n} x^n}{\pi^{2n}} \sum_{k=0}^{n} \frac{(2n - 2k + 1)_{2k}}{(2k)! (4n - 2k + 1)_{2k}} \left( -\frac{\pi^2}{x} \right)^k.
\]

Changing now \( n - k \) by \( k \) we get
\[
v_n(x) = (-1)^n 2^{2n} (4n - 1)!! \sum_{k=0}^{n} \frac{(2k + 1)_{2n-2k}}{(2n - 2k)! (2n + 2k + 1)_{2n-2k}} \left( -\frac{x}{\pi^2} \right)^k
\]

\[
= (-1)^n \sum_{k=0}^{n} \frac{(2n + 2k)!}{(2k)! (2n - 2k)!} \left( -\frac{x}{\pi^2} \right)^k.
\]

Since ([4], Chapter 7)
\[
J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x,
\]
then
\[
j_{-1/2,k} = (2k + 1) \frac{\pi}{2}, \quad k = 0, \pm 1, \pm 2, \ldots
\]

Therefore, the discrete orthogonal relation (8) for the sequence of polynomials \( \{v_n(x)\} \) becomes
\[
\sum_{k=0}^{\infty} v_n \left( \frac{1}{(2k + 1)^2} \right) v_m \left( \frac{1}{(2k + 1)^2} \right) \cdot \frac{1}{(2k + 1)^2} = \frac{\pi^2}{8 (4n + 1)} \delta_{mn}.
\]

For the coefficient of \( x^n \) in \( v_n(x) \) is \( \frac{(4n)!}{(2n)! \pi^{2n}} \) the orthogonal relation (12) can be rewritten in the form
\[
\sum_{k=0}^{\infty} v_n \left( \frac{1}{(2k + 1)^2} \right) \cdot \frac{1}{(2k + 1)^{2m+2}} = \frac{(2n)! \pi^{2n+2}}{8 (4n + 1)!} \delta_{mn}, \quad 0 \leq m \leq n.
\]
Putting the explicit representation (11) of \( v_n(x) \) into (13) we get

\[
\sum_{k=0}^{\infty} \left[ \sum_{l=0}^{n} \frac{(-1)^{n+l} (2n + 2l)!}{(2l)! (2n - 2l)! \pi^{2l}} \left( \frac{1}{(2k + 1)^2} \right)^l \right] \frac{1}{(2k + 1)^{2m+2}} = \frac{(2n)! \pi^{2n+2}}{8 (4n + 1)!} \delta_{mn}, \quad 0 \leq m \leq n. \tag{14}
\]

Changing the order of summation in (14) we obtain

\[
\sum_{l=0}^{n} \frac{(-1)^{n+l} (2n + 2l)!}{(2l)! (2n - 2l)! \pi^{2l}} \sum_{k=0}^{\infty} \frac{1}{(2k + 1)^{2m+2l+2}} = \frac{(2n)! \pi^{2n+2}}{8 (4n + 1)!} \delta_{mn}, \quad 0 \leq m \leq n.
\]

Applying the formula ([3], Chapter 1, § 1.13)

\[
\sum_{k=0}^{\infty} \frac{1}{(2k + 1)^{2m+2l+2}} = \frac{(-1)^{m+l} \pi^{2m+2l+2} (2^{2m+2l+2} - 1)}{2 (2m + 2l)!} B_{2m+2l+2}, \tag{15}
\]

we get

\[
\sum_{l=0}^{n} \frac{(2n + 2l) \left( 1 - 2^{2n+2l+2} \right)! B_{2m+2l+2}}{(2l)! (2n - 2l)! (2m + 2l + 2)!} = -\frac{(2n)!}{(4n + 1)! 4} \delta_{mn}, \quad 0 \leq m \leq n. \tag{16}
\]

Hence formula (9) is proved in case \( \sigma = 0 \).

We consider now the case \( \sigma = 1 \). Let

\[
t_n(x) = x^{-1} [v_{n+1}(x) + v_n(x)]. \tag{17}
\]

Since \( v_n(0) = (-1)^n \), it is easy to see that \( t_n(x) \) is a polynomial of precise degree \( n \). After replacing \( v_n(x) \) and \( v_{n+1}(x) \) in (17) by their explicit representation formulas (11) we obtain

\[
t_n(x) = \frac{2 (-1)^n (4n + 3)}{\pi^2} \sum_{k=0}^{n} \frac{(2n + 2k + 2)!}{(2k + 1)! (2n - 2k)!} \left( \frac{-x}{\pi^2} \right)^k. \tag{18}
\]
Using formula (12) twice we have

\[
\sum_{k=0}^{\infty} t_n \left( \frac{1}{(2k+1)^2} \right) \frac{1}{(2k+1)^{2m+4}} = \sum_{k=0}^{\infty} v_{n+1} \left( \frac{1}{(2k+1)^2} \right) \frac{1}{(2k+1)^{2m+2}} \\
+ \sum_{k=0}^{\infty} v_n \left( \frac{1}{(2k+1)^2} \right) \frac{1}{(2k+1)^{2m+2}} = \sum_{k=0}^{\infty} v_n \left( \frac{1}{(2k+1)^2} \right) \frac{1}{(2k+1)^{2m+2}}
\]

\[
= \frac{(2n)!\pi^{2n+2}}{8(4n+1)!} \delta_{mn}, \quad 0 \leq m \leq n. \quad (19)
\]

Putting expression (18) into (19) we get

\[
\sum_{k=0}^{\infty} \left[ \sum_{l=0}^{n} \frac{(-1)^{n+l} (2n+2l+2)!}{(2l+1)! (2n-2l)! \pi^{2l} (2k+1)^2} \right] \frac{1}{(2k+1)^{2m+4}} = \frac{(2n+1)!\pi^{2n+4}}{8(4n+3)!} \delta_{mn}, \quad 0 \leq m \leq n. \quad (20)
\]

Changing the order of summation in (20) we have

\[
\sum_{l=0}^{n} (-1)^{n+l} \frac{(2n+2l+2)!}{(2l+1)! (2n-2l)! (2n-2l+1)! \pi^{2l}} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^{2m+2l+4}} = \frac{(2n+1)!\pi^{2n+4}}{8(4n+3)!}, \quad 0 \leq m \leq n. \quad (21)
\]

Replacing the infinite sum in (21) by the formula (15) we obtain

\[
\sum_{l=0}^{n} \frac{(2n+2l+2)!}{(2l)! (2n-2l)! (2m+2l+4)!} \left( 1 - 2^{2m+2l+4} \right) B_{2m+2l+4} = \frac{(2n+1)!}{4(4n+3)!} \delta_{mn}, \quad 0 \leq m \leq n. \quad (22)
\]

Hence, the formula (9) is proved for the case \(\sigma = 1\).

3. The relation between Legendre and Euler polynomials

Now we shall use the formula of finite summation (9) to prove the following expansion theorem.
Theorem. Let \( \{P_n(x)\} \) be Legendre polynomial sequence and \( \{E_n(x)\} \) be Euler polynomial sequence. The following equality holds

\[
P_{2n+\sigma}(2x-1) = \sum_{l=0}^{n} \frac{(2n+2l+2\sigma)!}{(2l)!(2l+\sigma)!(2n-2l)!} E_{2l+\sigma}(x),
\]

\((n \geq 0; \ \sigma = 0 \text{ or } 1)\).

Proof. First, we apply the lemma in case \( \sigma = 0 \). In formula (16) replacing \( B_{2m+2l+2} \) by formula (6) we get

\[
\int_0^1 \frac{E_{2m}(x)}{(2m)!} \sum_{l=0}^{n} \frac{(2n+2l)!E_{2l}(x)}{[(2l)!]^2(2n-2l)!} = \frac{(2n)!\delta_{mn}}{(4n+1)!}, \quad 0 \leq m \leq n.
\]

If we set

\[
\sum_{l=0}^{n} \frac{(2n+2l)!E_{2l}(x)}{[(2l)!]^2(2n-2l)!} = P_{2n}^*(x),
\]

then \( P_{2n}^*(x) \) is a polynomial of degree \( 2n \), and we obtain

\[
\int_0^1 E_{2m}(x) P_{2n}^*(x) \, dx = \frac{\{(2n)!\}^2}{(4n+1)!} \delta_{mn}, \quad 0 \leq m \leq n.
\]

Using (6) again and noticing that \( B_{2m+1} = 0 \) for \( m > 0 \), we have

\[
\int_0^1 E_{2m+1}(x) P_{2n}^*(x) \, dx = \int_0^1 E_{2m+1}(x) \sum_{l=0}^{n} \frac{(2n+2l)!E_{2l}(x)}{[(2l)!]^2(2n-2l)!} \, dx
\]

\[
= \sum_{l=0}^{n} \frac{(2n+2l)!}{[(2l)!]^2(2n-2l)!} \int_0^1 E_{2m+1}(x) E_{2l}(x) \, dx
\]

\[
= \sum_{l=0}^{n} 4 \frac{(2n+2l)!}{(2l)!}(2m+1)! \frac{(2m+2l+3)! - 1}{(2m+2l+3)!} B_{2m+2l+3} = 0.
\]

From (26) and (27) we can conclude that the polynomial \( P_{2n}^*(x) \) is orthogonal to the system \( \{E_0(x), E_1(x), ..., E_{2n-1}(x)\} \) with respect to the weight 1 on the interval \([0,1]\). Since the system of Legendre polynomials \( P_n(2x-1) \) is also orthogonal with respect to the weight 1 on \([0,1]\), then there exists a sequence of scalars \( \{\alpha_n\} \) such that

\[
P_{2n}^*(x) = \alpha_n P_{2n}(2x-1), \quad n \geq 0.
\]

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The coefficients $\alpha_n$ can be found exactly. Indeed, identifying the coefficients of $x^{2n}$ in two polynomials in (28) we get $\alpha_n = 1$ and the proof of the theorem in case $\sigma = 0$ is finished.

For the case $\sigma = 1$ we shall apply the lemma for the corresponding $\sigma$. The method used here is similar to the previous one.

Again replacing $B_{2m+2l+4}$ in (22) by formula (6) we get

$$\int_0^1 \frac{E_{2m+1}(x)}{(2m+1)!} \sum_{l=0}^n \frac{(2n+2l+2)!E_{2l+1}(x)}{(2l)!(2l+1)!(2n-2l)!} dx = \frac{(2n+1)!}{(4n+3)!}\delta_{mn}, \quad 0 \leq m \leq n. \quad (29)$$

Setting

$$\sum_{l=0}^n \frac{(2n+2l+2)!E_{2l+1}(x)}{(2l)!(2l+1)!(2n-2l)!} = P^*_{2n+1}(x), \quad (30)$$

then $P^*_{2n+1}(x)$ is a polynomial of precise degree $2n+1$, and we have

$$\int_0^1 E_{2m+1}(x) P^*_{2n+1}(x) dx = \frac{(2n+1)!}{(4n+3)!}\delta_{mn}, \quad 0 \leq m \leq n. \quad (31)$$

Completely similar to the corresponding step in the proof of the theorem in case $\sigma = 0$ we also obtain

$$\int_0^1 E_{2m}(x) P^*_{2n+1}(x) dx = 0, \quad 0 \leq m \leq n. \quad (32)$$

Now, from (31), (32) and reasoning the same as before we can conclude that there exists a sequence of scalars $\{\beta_n\}$ with

$$P^*_{2n+1}(x) = \beta_n P_{2n+1}(2x - 1), \quad n \geq 0. \quad (33)$$

Comparing the coefficients of $x^{2n+1}$ we can find out the value of $\beta_n$ to be 1, and the proof of the theorem is finished.
References


