SOLUTION OF A FRACTIONAL DIFFERINTEGRAL EQUATION

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Abstract

We aim to extend the use of the Riemann-Liouville definition of fractional calculus to solve a differintegral equation of Volterra's type of the form $D^{\mu}f(x)+aD^{-\nu}f(x) = g(x)$, with positive $\Re(\mu)$ and $\Re(\nu)$, $a \in C$ and $g(x)$ being a given function. The solution is expressed in terms of the Mittag-Leffler functions and the approach used in the paper allows us to extend the range of the parameters μ and ν to a wider range than in Al-Saqabi [7].

Keywords: Riemann-Liouville fractional operators, fractional differintegral equation, Cauchy problem.

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1. INTRODUCTION

Among several definitions of fractional calculus, see, for example, Oldham and Spanier [1]; Nishimoto [2]; Samko, Kilbas and Marichev [3], and Miller and Ross [4], the Riemann-Liouville definition is the most widely used.

The Riemann-Liouville (R.-L.) fractional integral operator is defined by

$$D^{-\delta}f(x) = \frac{1}{\Gamma(\delta)}\int_0^x (x-t)^{\delta-1}f(t)dt, \quad \Re(\delta),\, x>0, \tag{1}$$

whereas the R.-L. fractional differential operator is defined by

$$D^{\alpha}f(x) = D^n\left[D^{\alpha-n}f(x)\right], \quad \Re(\alpha)\geq 0,\ x>0,\ n=[\alpha]+1 \tag{2}$$

(see K. Miller and B. Ross [4]).

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Our aim is to solve a general differintegral equation of Volterra's type of the form

$$D^{\mu}f(x)+\frac{a}{\Gamma(\nu)}\int_0^x(x-t)^{\nu-1}f(t)dt=g(x), \tag{3}$$

that can be rewritten in the R.-L. notations

$$D^{\mu}f(x)+aD^{-\nu}f(x)=g(x), \tag{4}$$

where $\Re(\mu)$ and $\Re(\nu)$ are positive, $g(x)$ is any given integrable function on the finite interval $[0,b]$ and $a\in C$.

Ross and Sachdeva [5], Suarez [6] have considered equation (3) for $\mu=0$, $a=1$ and ν being a positive rational number by applying successively the R.-L. operators until the integral equation reduced to an ordinary differential equation. Al-Saqabi [7] has generalized the method used in [6] and showed its efficiency for solving a differintegral equation (4) with $a=1$ and $\mu+\nu$ being a positive rational number. For the case $\Re(\mu)$ and $\Re(\nu)$ having different signs, equation (4) reduces to purely fractional integral or differential equations which were studied earlier by many other authors (see [3], chapter 8).

2. MAIN RESULTS

Applying the fractional integral operator $D^{-\mu}$ to the both sides of (4), we have

$$D^{-\mu}D^{\mu}f(x)+aD^{-\mu-\nu}f(x)=D^{-\mu}g(x). \tag{5}$$

Let n be an integer such that $n=[\mu]+1$. Then according to formula (2.60) from [3], we have

$$D^{-\mu}D^{\mu}f(x)=f(x)-\sum_{k=0}^{n-1}\frac{x^{\mu-k-1}}{\Gamma(\mu-k)}f_{n-\mu}^{(n-k-1)}(0), \tag{6}$$

where

$$f_{n-\mu}^{(n-k-1)}(x)=D^{n-k-1}D^{\mu-n}f(x)=D^{\mu-k-1}f(x). \tag{7}$$

Therefore (5) becomes

$$f(x)=D^{-\mu}g(x)-aD^{-\mu-\nu}f(x)+\sum_{k=0}^{n-1}\frac{x^{\mu-k-1}}{\Gamma(\mu-k)}D^{\mu-k-1}f(0). \tag{8}$$

Applying to the both sides of (8) operator $(-a)^mD^{-m(\mu+\nu)}$ we get

$$\begin{aligned}(-a)^mD^{-m(\mu+\nu)}f(x) &= (-a)^mD^{-\mu-m(\mu+\nu)}g(x)\\ +(-a)^{m+1}D^{-(m+1)(\mu+\nu)}f(x) &+ \sum_{k=0}^{n-1}D^{\mu-k-1}f(0)(-a)^mD^{-m(\mu+\nu)}\frac{x^{\mu-k-1}}{\Gamma(\mu-k)},\\ & m=0,1,2,\dots\ . \end{aligned} \tag{9}$$

Using equation (2.44) from [3]

$$D^{-m(\mu+\nu)}\frac{x^{\mu-k-1}}{\Gamma(\mu-k)}=\frac{x^{\mu+m(\mu+\nu)-k-1}}{\Gamma(\mu+m(\mu+\nu)-k)} \tag{10}$$

we have

$$\begin{aligned}(-a)^m D^{-m(\mu+\nu)}f(x) &= (-a)^m D^{-\mu-m(\mu+\nu)}g(x)\\ +(-a)^{m+1}D^{-(m+1)(\mu+\nu)}f(x) &+ \sum_{k=0}^{n-1} D^{\mu-k-1}f(0)(-a)^m\frac{x^{\mu+m(\mu+\nu)-k-1}}{\Gamma(\mu+m(\mu+\nu)-k)}.\end{aligned} \tag{11}$$

Summing up (11) from $m=0$ to ∞, we get

$$\begin{aligned}\sum_{m=0}^{\infty}(-a)^m D^{-m(\mu+\nu)}f(x) &= \sum_{m=0}^{\infty}(-a)^m D^{-\mu-m(\mu+\nu)}g(x)\\ +\sum_{m=1}^{\infty}(-a)^m D^{-m(\mu+\nu)}f(x) &+ \sum_{k=0}^{n-1} D^{\mu-k-1}f(0)x^{\mu-k-1}\sum_{m=0}^{\infty}\frac{(-ax^{\mu+\nu})^m}{\Gamma(\mu+m(\mu+\nu)-k)}.\end{aligned} \tag{12}$$

The inner series in formula (12) can be expressed via the Mittag-Leffler function. The Mittag-Leffler function

$$E_{\alpha,\beta}(x)=\sum_{m=0}^{\infty}\frac{x^m}{\Gamma(\alpha m+\beta)} \tag{13}$$

is defined usually under the restriction that α and β are real numbers and $\alpha>0$ (see Erdelyi [8]). But using the Stirling's asymptotic formula for the gamma function it is not difficult to see that the series in the right-hand side of formula (13) converges even when α,β are complex numbers and $\Re(\alpha)>0$. Furthermore, the resulting function is also an entire function and has many properties of the Mittag-Leffler function (in particular, formula (23) and (25)) are still valid.
Canceling all common terms in the left and right-hand sides of (12) and rewriting the inner series as the Mittag-Leffler function in the general meaning we get

$$\begin{aligned}f(x) &= \sum_{m=0}^{\infty}(-a)^m D^{-\mu-m(\mu+\nu)}g(x)\\ &+ \sum_{k=0}^{n-1} D^{\mu-k-1}f(0)x^{\mu-k-1}E_{\mu+\nu,\mu-k}\left(-ax^{\mu+\nu}\right).\end{aligned} \tag{14}$$

We have

$$\begin{aligned}\sum_{m=0}^{\infty}(-a)^m D^{-\mu-m(\mu+\nu)}g(x) &= \sum_{m=0}^{\infty}\int_0^x\frac{(x-t)^{\mu+m(\mu+\nu)-1}}{\Gamma(\mu+m(\mu+\nu))}(-a)^m g(t)dt\\ &= \int_0^x (x-t)^{\mu-1}E_{\mu+\nu,\mu}\left(-a(x-t)^{\mu+\nu}\right)g(t)dt,\end{aligned}$$

where the interchange of order of summation and integration is possible, since

$$\sum_{m=0}^{\infty}\left|\frac{(x-t)^{m(\mu+\nu)}}{\Gamma(\mu+m(\mu+\nu))}\right|$$

is uniformly bounded in the domain $0 \le t \le x \le b$. Therefore, formula (14) now becomes

$$f(x) = \int_0^x (x-t)^{\mu-1} E_{\mu+\nu,\mu}\left(-a(x-t)^{\mu+\nu}\right) g(t)dt + \sum_{k=0}^{n-1} \alpha_k x^{\mu-k-1} E_{\mu+\nu,\mu-k}\left(-ax^{\mu+\nu}\right). \qquad (15)$$

We shall prove that formula (15) gives indeed a general solution of equation (4) when $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$ are arbitrary numbers. Applying operator $aD^{-\nu}$ to the both sides of formula (15), or equivalently, to (14), we have

$$\begin{aligned} aD^{-\nu}f(x) &= a\sum_{m=0}^{\infty}(-a)^m D^{-(\mu+\nu)(m+1)}g(x) + a\sum_{k=0}^{n-1}\alpha_k \sum_{m=0}^{\infty}(-a)^m D^{-\nu}\frac{x^{\mu+m(\mu+\nu)-k-1}}{\Gamma(\mu+m(\mu+\nu)-k)} \\ &= -\sum_{m=1}^{\infty}(-a)^m D^{-(\mu+\nu)m}g(x) - \sum_{k=0}^{n-1}\alpha_k \sum_{m=0}^{\infty}(-a)^{m+1}\frac{x^{(\mu+\nu)(m+1)-k-1}}{\Gamma((\mu+\nu)(m+1)-k)} \\ &= -\sum_{m=1}^{\infty}(-a)^m D^{-(\mu+\nu)m}g(x) - \sum_{k=0}^{n-1}\alpha_k \sum_{m=1}^{\infty}(-a)^{m}\frac{x^{(\mu+\nu)m-k-1}}{\Gamma((\mu+\nu)m-k)}. \end{aligned} \qquad (16)$$

On the other hand,

$$\begin{aligned} D^{\mu}f(x) &= \sum_{m=0}^{\infty}(-a)^m D^{\mu}D^{-\mu-m(\mu+\nu)}g(x) \\ &+ \sum_{k=0}^{n-1}\alpha_k \sum_{m=0}^{\infty}(-a)^m D^{\mu}\frac{x^{\mu+m(\mu+\nu)-k-1}}{\Gamma(\mu+m(\mu+\nu)-k)}. \end{aligned} \qquad (17)$$

When $m = 0$, we get

$$D^{\mu}x^{\mu-k-1} = 0 \quad \text{for} \quad k = 0, 1, \ldots, n-1, \qquad (18)$$

whereas when $m > 0$

$$D^{\mu}\frac{x^{\mu+m(\mu+\nu)-k-1}}{\Gamma(\mu+m(\mu+\nu)-k)} = \frac{x^{m(\mu+\nu)-k-1}}{\Gamma(m(\mu+\nu)-k)} \qquad (19)$$

(see equation (2.35) from [3]). Consequently

$$D^{\mu}f(x) = \sum_{m=0}^{\infty}(-a)^m D^{-m(\mu+\nu)}g(x) + \sum_{k=0}^{n-1}\alpha_k \sum_{m=1}^{\infty}(-a)^m \frac{x^{m(\mu+\nu)-k-1}}{\Gamma(m(\mu+\nu)-k)}. \qquad (20)$$

Now summing up formulae (16) and (20), we obtain

$$D^{\mu}f(x) + aD^{-\nu}f(x) = g(x), \qquad (21)$$

that means (15) is a solution of equation (4). Consequently, the homogeneous equation (4) has $[\mu]+1$ independent solutions. If we consider a Cauchy problem

$$D^{\mu}f(x) + aD^{-\nu}f(x) = g(x), \quad D^{\mu-k-1}f(0) = \alpha_k,\ k = 0, 1, \ldots, [\mu], \qquad (22)$$

then the Cauchy problem (22) has the unique solution (15).

3. PARTICULAR CASES

(i) Let $g(t) = t^{\alpha-1},\ \alpha > 0$. By employing equation (6) from [9]:

$$\int_0^x (x-t)^{\alpha-1} E_{\mu+\nu,\mu}(-at^{\mu+\nu}) t^{\mu-1} dt = \Gamma(\alpha) x^{\mu+\alpha-1} E_{\mu+\nu,\mu+\alpha}(-ax^{\mu+\nu}), \tag{23}$$

the solution $f(x)$ will be given in this case as follows:

$$f(x) = \Gamma(\alpha) x^{\mu+\alpha-1} E_{\mu+\nu,\mu+\alpha}(-ax^{\mu+\nu}) + \sum_{k=0}^{n-1} \alpha_k x^{\mu-k-1} E_{\mu+\nu,\mu-k}(-ax^{\mu+\nu}). \tag{24}$$

(ii) Put $g(t) = E_{\mu+\nu,\alpha+\nu}(-bt^{\mu+\nu}) t^{\alpha+\nu-1}$. By employing equation (10) from [9]:

$$\int_0^x E_{\mu+\nu,\alpha+\nu}(zt^{\mu+\nu}) t^{\alpha+\nu-1} E_{\mu+\nu,\mu}(-a(x-t)^{\mu+\nu})(x-t)^{\mu-1} dt$$
$$= \frac{E_{\mu+\nu,\alpha}(zx^{\mu+\nu}) - E_{\mu+\nu,\alpha}(-ax^{\mu+\nu})}{z+a} x^{\alpha-1} \tag{25}$$

the solution $f(x)$ in this case becomes

$$f(x) = \frac{E_{\mu+\nu,\alpha}(-bx^{\mu+\nu}) - E_{\mu+\nu,\alpha}(-ax^{\mu+\nu})}{a-b} x^{\alpha-1}$$
$$+ \sum_{k=0}^{n-1} \alpha_k x^{\mu-k-1} E_{\mu+\nu,\mu-k}(-ax^{\mu+\nu}). \tag{26}$$

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