Asymptotic Formulas for Generalized Elliptic-type Integrals

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Abstract

Epstein-Hubbell [6] elliptic-type integrals occur in radiation field problems. The object of the present paper is to consider a unified form of different elliptic-type integrals, defined and developed recently by several authors. We obtain asymptotic formulas for the generalized elliptic-type integrals.

Keywords—Elliptic-type Integrals, Hypergeometric Functions, Asymptotic Formulas.

1. INTRODUCTION

Elliptic integrals occur in a number of physical problems [1,5,7,17], and frequently in the form of multiple integrals. One of the integrals being performed, it leads to an integrand which itself involves elliptic integrals.

Epstein and Hubbell [6] have treated a family of elliptic-type integrals

$$\Omega_j(k) = \int_0^\pi (1 - k^2 \cos\theta)^{-j-1/2} d\theta, \quad 0 \le k < 1,\ j = 0, 1, 2, \dots \tag{1}$$

Certain problems dealing with the computation of the radiation field off axis from a uniform circular disc radiating according to an arbitrary angular distribution law [2,10], when treated with a Legendre polynomial expansion method, give rise to integrals of form (1). For $j = 0, 1$ formula (1) reduces to

$$\Omega_0(k) = \frac{\sqrt{2}\lambda}{k} K(\lambda), \tag{2}$$

and

$$\Omega_1(k) = \frac{\sqrt{2}\lambda}{k(1-k^2)} E(\lambda), \tag{3}$$

*CIMA, University of Zulia, Maracaibo, Venezuela where $\lambda^2 = \frac{2k^2}{1+k^2}$, and $K(\lambda)$ and $E(\lambda)$ are the complete elliptic integrals of the first and second kinds, respectively [1].

Elliptic integral (1) has been generalized and studied by several authors, Kalla [11], Kalla, Conde and Hubbell [12], Kalla and Al-Saqabi [14,15,16], Srivastava et al [20,21], and others [3,4,19,22].

Kalla et al [12,13] and Glasser and Kalla [9] have developed a systematic study of the following family of elliptic-type integrals

$$R_\mu(k,\alpha,\gamma) = \int_0^\pi \frac{\cos^{2\alpha-1}(\theta/2)\sin^{2\gamma-2\alpha-1}(\theta/2)}{(1-k^2\cos\theta)^{\mu+1/2}}d\theta,$$
$$0 \le k < 1,\ \Re(\gamma) > \Re(\alpha) > 0,\ \Re(\mu) > -\frac{1}{2}. \tag{4}$$

It can be easily verified that

$$R_j(k,\frac{1}{2},1) = \Omega_j(k), \tag{5}$$

and in terms of hypergeometric functions

$$R_\mu(k,\alpha,\gamma) = \frac{\Gamma(\alpha)\Gamma(\gamma-\alpha)}{\Gamma(\gamma)(1+k^2)^{\mu+1/2}}F\left(\alpha,\mu+\frac{1}{2};\gamma;\frac{2k^2}{1+k^2}\right). \tag{6}$$

These authors have obtained a number of recurrence formulas, asymptotic expansion for $k^2 \to 1$, computer algorithms, integrals and other useful properties.

Recently, Srivastava and Siddiqi [20] have given a unified presentation of certain families of elliptic-type integrals in the following form

$$\Lambda_{\lambda,\mu}^{(\alpha,\beta)}(\rho;k) = \int_0^\pi \frac{\cos^{2\alpha-1}(\theta/2)\sin^{2\beta-1}(\theta/2)}{(1-k^2\cos\theta)^{\mu+1/2}}[1-\rho\sin^2(\theta/2)]^{-\lambda}d\theta,$$
$$0 \le k < 1,\ \Re(\alpha),\Re(\beta) > 0;\ \lambda,\mu \in C;\ |\rho| < 1. \tag{7}$$

By comparing (6) and (7) we have

$$\Lambda_{0,\mu}^{(\alpha,\gamma-\alpha)}(\rho;k) = \Lambda_{\lambda,\mu}^{(\alpha,\gamma-\alpha)}(0;k) = R_\mu(k,\alpha,\gamma). \tag{8}$$

Here we consider an another generalization of the elliptic-type integrals

$$\Lambda_{(\lambda,\gamma,\mu)}^{(\alpha,\beta)}(\rho,\delta;k) = \int_0^\pi \frac{\cos^{2\alpha-1}(\theta/2)\sin^{2\beta-1}(\theta/2)}{(1-k^2\cos\theta)^{\mu+1/2}}[1-\rho\sin^2(\theta/2)]^{-\lambda}[1+\delta\cos^2(\theta/2)]^{-\gamma}d\theta,$$
$$0 \le k < 1,\ \Re(\alpha),\Re(\beta) > 0;\ \lambda,\mu,\gamma \in C; \tag{9}$$
$$\text{either}\quad |\rho|,|\delta| < 1 \text{ or } \rho, (\text{or } \delta) \in C \ \text{ whenever } \lambda = -m \ (\text{or } \gamma = -m),\ m \in N_0.$$

We observe that

$$\Lambda_{(\lambda,\gamma,\mu)}^{(\alpha,\beta)}(\rho,0;k) = \Lambda_{(\lambda,0,\mu)}^{(\alpha,\beta)}(\rho,\delta;k) = \Lambda_{\lambda,\mu}^{(\alpha,\beta)}(\rho;k), \tag{10}$$

where $\Lambda_{\lambda,\mu}^{(\alpha,\beta)}(\rho;k)$ is defined by (7), and contains other families of elliptic-type integrals as its special cases.

In this paper, first we express our generalized elliptic-type integral (9) in terms of the Lauricella's hypergeometric function of three variables $F_D^{(3)}$ and then obtain its asymptotic expansion as $k^2 \to 1$. Corresponding special cases for $R_\mu(k,\alpha,\gamma)$ and $\Omega_j(k)$ are also considered.

2. ASYMPTOTIC EXPANSION FOR $\Lambda_{(\lambda,\gamma,\mu)}^{(\alpha,\beta)}(\rho,\delta;k)$

From definition (9) we have

$$\begin{aligned}\Lambda_{(\lambda,\gamma,\mu)}^{(\alpha,\beta)}(\rho,\delta;k) &= (1+\delta)^{-\gamma}(1-k^2)^{-\mu-1/2} \\ &\int_0^1 \omega^{\beta-1}(1-\omega)^{\alpha-1}(1-\rho\omega)^{-\lambda}\left(1-\frac{\delta\omega}{1+\delta}\right)^{-\gamma}\left(1-\frac{2k^2\omega}{k^2-1}\right)^{-\mu-1/2} d\omega.\end{aligned} \tag{11}$$

Comparing the integral in (11) with the integral representation of the Lauricella's hypergeometric function of three variables $F_D^{(3)}$ (see [8]) we can express our generalized elliptic-type integral (9) in terms of the Lauricella's hypergeometric function $F_D^{(3)}$ as follows

$$\begin{aligned}\Lambda_{(\lambda,\gamma,\mu)}^{(\alpha,\beta)}(\rho,\delta;k) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}(1+\delta)^{-\gamma}(1-k^2)^{-\mu-1/2} \\ F_D^{(3)}\left(\beta,\lambda,\gamma,\mu+\frac{1}{2};\alpha+\beta;\rho;\frac{\delta}{1+\delta},\frac{2k^2}{k^2-1}\right).\end{aligned} \tag{12}$$

To obtain asymptotic expansion of $\Lambda_{(\lambda,\gamma,\mu)}^{(\alpha,\beta)}(\rho,\delta;k)$ as $k^2 \to 1$ we express the Lauricella's hypergeometric function $F_D^{(3)}$ in (12) as a double series of the Gauss hypergeometric functions

$$\begin{aligned}F_D^{(3)}\left(\beta,\lambda,\gamma,\mu+\frac{1}{2};\alpha+\beta;\rho;\frac{\delta}{1+\delta},\frac{2k^2}{k^2-1}\right) = \sum_{m,n=0}^{\infty}\frac{(\beta)_{m+n}(\lambda)_m(\gamma)_n}{(\alpha+\beta)_{m+n}m!n!}\rho^m\left(\frac{\delta}{1+\delta}\right)^n \\ {}_2F_1\left(\beta+m+n,\mu+1/2;\alpha+\beta+m+n;\frac{2k^2}{k^2-1}\right).\end{aligned} \tag{13}$$

Using the analytic continuation formula 15.3.7 [1] for the Gauss hypergeometric functions in (13) we get

$$\begin{aligned}F_D^{(3)}\left(\beta,\lambda,\gamma,\mu+\frac{1}{2};\alpha+\beta;\rho;\frac{\delta}{1+\delta},\frac{2k^2}{k^2-1}\right) = \frac{\Gamma(\alpha+\beta)\Gamma(\mu-\beta+1/2)}{\Gamma(\alpha)\Gamma(\mu+1/2)}\left(\frac{1-k^2}{2k^2}\right)^\beta \\ \sum_{m,n=0}^{\infty}\frac{(\beta)_{m+n}(\lambda)_m(\gamma)_n}{(1/2-\mu+\beta)_{m+n}m!n!}\left(\frac{\rho(k^2-1)}{2k^2}\right)^m\left(\frac{\delta(k^2-1)}{2k^2(1+\delta)}\right)^n\end{aligned}$$

$$
{}_2F_1\left(\beta+m+n, 1-\alpha; 1/2-\mu+\beta+m+n; \frac{k^2-1}{2k^2}\right)
$$

$$
+\frac{\Gamma(\alpha+\beta)\Gamma(\beta-\mu-1/2)}{\Gamma(\beta)\Gamma(\alpha+\beta-\mu-1/2)}\left(\frac{1-k^2}{2k^2}\right)^{\mu+1/2}\sum_{m,n=0}^{\infty}\frac{(\beta-\mu-1/2)_{m+n}(\lambda)_m(\gamma)_n}{(\alpha+\beta-\mu-1/2)_{m+n}m!n!}
$$

$$
(\rho)^m\left(\frac{\delta}{1+\delta}\right)^n {}_2F_1\left(\mu+1/2, \mu-\alpha-\beta-m-n+3/2; \mu-\beta-m-n+3/2; \frac{1-k^2}{2k^2}\right), \quad (14)
$$

if $\mu-\beta+1/2$ is not an integer. Therefore,

$$
\Lambda^{(\alpha,\beta)}_{(\lambda,\gamma,\mu)}(\rho,\delta;k)=\frac{\Gamma(\beta)\Gamma(\mu-\beta+1/2)}{\Gamma(\mu+1/2)}2^{-\beta}k^{-2\beta}(1+\delta)^{-\gamma}(1-k^2)^{\beta-\mu-1/2}
$$

$$
F_D^{(3)}\left(\beta,\lambda,\gamma,1-\alpha;\beta-\mu+1/2;\frac{\rho(k^2-1)}{2k^2},\frac{\delta(k^2-1)}{2k^2(1+\delta)},\frac{k^2-1}{2k^2}\right)+\frac{\Gamma(\alpha)\Gamma(\beta-\mu-1/2)}{\Gamma(\alpha+\beta-\mu-1/2)}
$$

$$
2^{-\mu-1/2}k^{-2\mu-1}(1+\delta)^{-\gamma}\sum_{n=0}^{\infty}\frac{(\mu-\alpha-\beta+3/2)_n(\mu+1/2)_n}{(\mu-\beta+3/2)_n n!}\left(\frac{1-k^2}{2k^2}\right)^n
$$

$$
F_1\left(\beta-\mu-n-1/2,\lambda,\gamma;\alpha+\beta-\mu-n-1/2;\rho,\frac{\delta}{1+\delta}\right), \quad (15)
$$

where F_1 is the Appell hypergeometric function of two variables [8]. Formula (15) can be considered as the asymptotic series for $\Lambda^{(\alpha,\beta)}_{(\lambda,\gamma,\mu)}(\rho,\delta;k)$ as $k^2\to 1$ if $\mu-\beta+1/2$ is not an integer.

Let now $\mu-\beta+1/2$ be an integer, say $\mu-\beta+1/2=\pm l,\ l=0,1,2,\ldots$. First, let $\mu+1/2=\beta-l$. Applying formula 15.3.14 [1]

$$
{}_2F_1\left(\beta+m+n,\beta-l;\alpha+\beta+m+n;\frac{2k^2}{k^2-1}\right)
$$

$$
=\frac{\Gamma(\alpha+\beta+m+n)}{\Gamma(\beta+m+n)\Gamma(\alpha+m+n+l)}\left(\frac{1-k^2}{2k^2}\right)^{\beta+m+n}
$$

$$
\sum_{r=0}^{\infty}\frac{(\beta-l)_{r+m+n+l}(1-\alpha-m-n-l)_{r+m+n+l}}{(r+m+n+l)!r!}\left(\frac{k^2-1}{2k^2}\right)^r\Big[\ln(2k^2)
$$

$$
-\ln(1-k^2)+\Psi(1+m+n+l+r)+\Psi(1+r)-\Psi(\beta+m+n+r)-\Psi(\alpha-r)\Big]
$$

$$
+\left(\frac{1-k^2}{2k^2}\right)^{\beta-l}\frac{\Gamma(\alpha+\beta+m+n)}{\Gamma(\beta+m+n)}\sum_{r=0}^{l-1}\frac{(\beta-l)_r(m+n+l-r-1)!}{\Gamma(\alpha+m+n+l-r)r!}\left(\frac{1-k^2}{2k^2}\right)^r, \quad (16)
$$

we obtain

$$
\Lambda^{(\alpha,\beta)}_{(\lambda,\gamma,\beta-l-1/2)}(\rho,\delta;k)=2^{-\beta}k^{-2\beta}(1+\delta)^{-\gamma}(1-k^2)^l(1-\beta)_l
$$

$$
\sum_{m,n,r=0}^{\infty}\frac{(\beta)_{m+n+r}(\lambda)_m(\gamma)_n(1-\alpha)_r}{(m+n+l+r)!m!n!r!}\left(\frac{\rho(k^2-1)}{2k^2}\right)^m\left(\frac{\delta(k^2-1)}{2k^2}\right)^n\left(\frac{k^2-1}{2k^2}\right)^r
$$

$$
\Big[\ln(2k^2)-\ln(1-k^2)+\Psi(1+m+n+l+r)+\Psi(1+r)-\Psi(\beta+m+n+r)
$$

$$-\Psi(\alpha-r)]+(-1)^l 2^{l-\beta}k^{2l-2\beta}(1+\delta)^{-\gamma}(1-\beta)_l\sum_{r=0}^{l-1}\frac{(1-\alpha)_{l-r}(l-r-1)!}{(1-\beta)_{l-r}r!}\left(\frac{1-k^2}{2k^2}\right)^r$$
$$F_1\left(l-r,\lambda,\gamma;\alpha+l-r;\rho,\frac{\delta}{1+\delta}\right). \quad (17)$$

Let now $\mu+1/2=\beta+l$. If $m+n<l$ we have

$$ {}_2F_1\left(\beta+m+n,\beta+l;\alpha+\beta+m+n;\frac{2k^2}{k^2-1}\right)=\frac{\Gamma(\alpha+\beta+m+n)}{\Gamma(\alpha)\Gamma(\beta+l)}\left(\frac{1-k^2}{2k^2}\right)^{\beta+l}$$
$$\sum_{r=0}^{\infty}\frac{(\beta+m+n)_{r+l-m-n}(1-\alpha)_{r+l-m-n}}{(r+l-m-n)!r!}\left(\frac{k^2-1}{2k^2}\right)^r\Big[\ln(2k^2)-\ln(1-k^2)$$
$$+\Psi(1+r+l-m-n)+\Psi(1+r)-\Psi(\beta+r+l)-\Psi(\alpha+m+n-r-l)]$$
$$+\frac{\Gamma(\alpha+\beta+m+n)}{\Gamma(\beta+l)}\left(\frac{1-k^2}{2k^2}\right)^{\beta+m+n}$$
$$\sum_{r=0}^{l-m-n-1}\frac{(\beta+m+n)_r(l-m-n-r-1)!}{\Gamma(\alpha-r)r!}\left(\frac{k^2-1}{2k^2}\right)^r. \quad (18)$$

If $m+n\geq l$ then

$$ {}_2F_1\left(\beta+m+n,\beta+l;\alpha+\beta+m+n;\frac{2k^2}{k^2-1}\right)$$
$$\frac{\Gamma(\alpha+\beta+m+n)}{\Gamma(\beta+m+n)\Gamma(\alpha+m+n-l)}\left(\frac{1-k^2}{2k^2}\right)^{\beta+m+n}$$
$$\sum_{r=0}^{\infty}\frac{(\beta+l)_{m+n+r-l}(1-\alpha+l-m-n)_{m+n+r-l}}{(m+n+r-l)!r!}\left(\frac{k^2-1}{2k^2}\right)^r\Big[\ln(2k^2)$$
$$-\ln(1-k^2)+\Psi(1+m+n+r-l)+\Psi(1+r)-\Psi(\beta+m+n+r)-\Psi(\alpha-r)\Big]$$
$$+\frac{\Gamma(\alpha+\beta+m+n)}{\Gamma(\beta+m+n)}\left(\frac{1-k^2}{2k^2}\right)^{\beta+l}\sum_{r=0}^{m+n-l-1}\frac{(\beta+l)_r(m+n-l-r-1)!}{\Gamma(\alpha+m+n-l-r)r!}\left(\frac{k^2-1}{2k^2}\right)^r. \quad (19)$$

Therefore,

$$\Lambda^{(\alpha,\beta)}_{(\lambda,\gamma,\beta+l-1/2)}(\rho,\delta;k)=2^{-\beta-l}k^{-2\beta-2l}(1+\delta)^{-\gamma}$$
$$\sum_{r=0}^{\infty}\sum_{m+n<l}\frac{(1-\alpha)_{r+l-m-n}(\lambda)_m(\gamma)_n(\beta+l)_r}{(r+l-m-n)!m!n!r!}\rho^m\left(\frac{\delta}{1+\delta}\right)^n\left(\frac{k^2-1}{2k^2}\right)^r$$
$$\Big[\ln(2k^2)-\ln(1-k^2)+\Psi(1+r+l-m-n)+\Psi(1+r)-\Psi(\beta+r+l)$$
$$-\Psi(\alpha+m+n-r-l)]+2^{-\beta}k^{-2\beta}(1+\delta)^{-\gamma}(1-k^2)^{-l}\frac{1}{(\beta)_l}$$
$$\sum_{m+n<l}\sum_{r=0}^{l-m-n-1}\frac{(\beta)_{m+n+r}(\lambda)_m(\gamma)_n(1-\alpha)_r(l-m-n-r-1)!}{m!n!r!}$$

$$\left(\frac{\rho(1-k^2)}{2k^2}\right)^m \left(\frac{\delta(1-k^2)}{2k^2(1+\delta)}\right)^n \left(\frac{1-k^2}{2k^2}\right)^r + 2^{-\beta}k^{-2\beta}(1+\delta)^{-\gamma}(k^2-1)^{-l}\frac{1}{(\beta)_l}$$

$$\sum_{m+n\geq l}\sum_{r=0}^{\infty}\frac{(\beta)_{m+n+r}(\lambda)_m(\gamma)_n(1-\alpha)_r}{(m+n+r-l)!m!n!r!}\left(\frac{\rho(k^2-1)}{2k^2}\right)^m\left(\frac{\delta(k^2-1)}{2k^2(1+\delta)}\right)^n\left(\frac{k^2-1}{2k^2}\right)^r$$

$$\Big[\ln(2k^2)-\ln(1-k^2)$$

$$+\Psi(1+m+n+r-l)+\Psi(1+r)-\Psi(\beta+m+n+r)-\Psi(\alpha-r)\Big]$$

$$+2^{-\beta-l}k^{-2\beta-2l}(1+\delta)^{-\gamma}\sum_{m+n\geq l}\sum_{r=0}^{m+n-l-1}\frac{(\lambda)_m(\gamma)_n(\beta+l)_r(m+n-l-r-1)!}{(\alpha)_{m+n-r-l}!m!n!r!}$$

$$\rho^m\left(\frac{\delta}{1+\delta}\right)^n\left(\frac{k^2-1}{2k^2}\right)^r. \qquad (20)$$

3. ASYMPTOTIC EXPANSION FOR $R_\mu(k,\alpha,\gamma)$

Asymptotic expansion for $R_\mu(k,\alpha,\gamma)$ can be obtained in similar manner. Indeed, using the hypergeometric representation for $R_\mu(k,\alpha,\gamma)$

$$R_\mu(k,\alpha,\gamma)=\frac{\Gamma(\alpha)\Gamma(\gamma-\alpha)}{\Gamma(\gamma)}(1-k^2)^{-\mu-1/2}$$

$${}_2F_1\left(\gamma-\alpha,\mu+1/2;\gamma;\frac{2k^2}{k^2-1}\right), \qquad (21)$$

and formula 15.3.7 [1] to the hypergeometric function in (21) we have

$$R_\mu(k,\alpha,\gamma)=\frac{\Gamma(\gamma-\alpha)\Gamma(\alpha+\mu-\gamma+1/2)}{\Gamma(\mu+1/2)}2^{\alpha-\gamma}k^{2\alpha-2\gamma}(1-k^2)^{\gamma-\alpha-\mu-1/2}$$

$${}_2F_1\left(\gamma-\alpha,1-\alpha;\gamma-\alpha-\mu+1/2;\frac{k^2-1}{2k^2}\right)$$

$$+\frac{\Gamma(\alpha)\Gamma(\gamma-\alpha-\mu-1/2)}{\Gamma(\gamma-\mu-1/2)}2^{-\mu-1/2}k^{-2\mu-1}$$

$${}_2F_1\left(\mu+1/2,\mu-\gamma+3/2;\alpha+\mu-\gamma+3/2;\frac{k^2-1}{2k^2}\right), \qquad (22)$$

if $\gamma-\alpha-\mu+1/2$ is not an integer.

If $\mu=\gamma-\alpha+l-1/2, \quad l=0,1,2,\ldots$, we have

$$R_{\gamma-\alpha+l-1/2}(k,\alpha,\gamma)=\frac{(2k^2)^{\alpha-\gamma-l}}{(\gamma-\alpha)_l}\sum_{n=0}^{\infty}\frac{(\gamma-\alpha)_{n+l}(1-\alpha)_{n+l}}{(n+l)!n!}\left(\frac{k^2-1}{2k^2}\right)^n$$

$$\Big[\ln(2k^2)-\ln(1-k^2)+\Psi(1+n+l)+\Psi(1+n)-\Psi(\gamma-\alpha+n+l)-\Psi(\alpha-n-l)\Big]$$

$$+\left(\frac{1-k^2}{2k^2}\right)^{\gamma-\alpha}\sum_{n=0}^{l-1}\frac{(\gamma-\alpha)_n(1-\alpha)_n(l-n-1)!}{(\gamma-\alpha)_l n!}\left(\frac{1-k^2}{2k^2}\right)^n. \qquad (23)$$

If $\mu = \gamma - \alpha - l - 1/2, \quad l = 1, 2, \ldots$, we obtain

$$R_{\gamma-\alpha-l-1/2}(k,\alpha,\gamma) = \frac{(1+\alpha-\gamma)_l}{(\alpha)_l}\left(\frac{1-k^2}{2k^2}\right)^{\gamma-\alpha}\sum_{n=0}^{\infty}\frac{(\gamma-\alpha)_n(1-\alpha)_n}{(n+l)!n!}\left(\frac{k^2-1}{2k^2}\right)^n$$
$$\Big[\ln(2k^2) - \ln(1-k^2) + \Psi(1+n+l) + \Psi(1+n) - \Psi(\gamma-\alpha+l) - \Psi(\alpha-n)\Big]$$
$$+(2k^2)^{\alpha-\gamma+l}\sum_{n=0}^{l-1}\frac{(\gamma-\alpha-l)_n(l-n-1)!}{(\alpha)_{l-n}n!}\left(\frac{k^2-1}{2k^2}\right)^n. \quad (24)$$

4. SPECIAL CASES

From the general formulas established in the previous sections for $\Lambda^{(\alpha,\beta)}_{(\lambda,\gamma,\mu)}(\rho,\delta;k)$ and $R_\mu(k,\alpha,\gamma)$, one can derive the corresponding asymptotic formulas for other types of elliptic-type integrals by choosing suitable parameter.

For example, if one set $\alpha = 1/2$, $\gamma = 1$, and $\mu = j$ in $R_\mu(k,\alpha,\gamma)$, it reduces to Epstein-Hubbell elliptic-type integral $\Omega_j(k)$, and then we have

$$\Omega_j(k) = \frac{(2k^2)^{-j-1/2}}{(1/2)_j}\sum_{n=0}^{\infty}\frac{(1/2)_{j+n}(1/2)_{j+n}}{(n+j)!n!}\left(\frac{k^2-1}{2k^2}\right)^n$$
$$\Big[\ln(2k^2) - \ln(1-k^2) + \Psi(1+n+j) + \Psi(1+n) - \Psi(1/2+n+j) - \Psi(1/2-n-j)\Big]$$
$$+\left(\frac{1-k^2}{2k^2}\right)^{1/2}\sum_{n=0}^{j-1}\frac{(j-n-1)!(1/2)_n(1/2)_n}{(1/2)_j n!}\left(\frac{1-k^2}{2k^2}\right)^n. (25)$$

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