Multidimensional Fractional Calculus Operators
Involving the Gauss Hypergeometric Function

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Abstract

This paper deals with some multidimensional integral operators involving the Gauss
hypergeometric function in the kernel and generating the multidimensional modified
fractional calculus operators introduced in [8]. Some mapping properties, weighted
inequalities, a formula of integration by parts and index laws are obtained.

1. Introduction and Preliminaries

As usual C and R+ denote the sets of complex and non-negative real numbers, respec-
tively, and α, β, γ, · · · stand for complex numbers. Let R+n denote the set of n-tuple non-
negative real numbers, and Cn of n-tuple complex numbers. We reserve a, b, c, · · ·, x, · · · in
most cases for elements of Cn, that means x = (x1, · · · , xn) etc. We set xα = xα1 · · · xαn and
x.1 = x1 + · · · + xn.

In [8] the multidimensional modified fractional integrals of order α with Re(α) > 0 are
defined as follows:

\[
X_+^\alpha f(x) = \frac{1}{\Gamma(\alpha + 1)} \frac{\partial^n}{\partial x_1 \cdots \partial x_n} \int_{\mathbb{R}_+^n} \left[ \min \left\{ \frac{x_1}{t_1}, \cdots, \frac{x_n}{t_n} \right\} - 1 \right]_+^\alpha f(t) dt,
\]

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\[ X_\alpha f(x) = \frac{(-1)^n}{\Gamma(\alpha + 1)} \frac{\partial^n}{\partial x_1 \cdots \partial x_n} \int_{\mathbb{R}^n_+} \left[ 1 - \max \left\{ \frac{x_1}{t_1}, \ldots, \frac{x_n}{t_n} \right\} \right]^\alpha f(t) dt, \]

where \( \varphi_+(x) \) is defined from a real valued function \( \varphi(x) \) by

\[
\varphi_+(x) = \begin{cases} 
\varphi(x), & \text{if } \varphi(x) > 0 \\
0, & \text{if } \varphi(x) \leq 0.
\end{cases}
\]

They are generalizations of the one-dimensional Riemann-Liouville and Weyl fractional integral operators, respectively [6].

In [4] another kind of generalization of the Riemann-Liouville and Weyl fractional integral operators involving the Gauss hypergeometric function \( _2F_1(\alpha, \beta; \gamma; z) \) [2] is introduced

\[
I^\alpha,\beta,\eta_x f = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} _2F_1 \left( \alpha + \beta, \alpha + \eta; 1 + \alpha; 1 - \frac{t}{x} \right) f(t) dt,
\]

(1.3)

\[
J^\alpha,\beta,\eta_x f = \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} t^{-\alpha-\beta} _2F_1 \left( \alpha + \beta, -\eta; 1 + \alpha; 1 - \frac{x}{t} \right) f(t) dt.
\]

(1.4)

In this paper some multidimensional modified fractional integral operators \( S^\alpha,\beta,\eta_+; n \) and \( S^\alpha,\beta,\eta_-; n \) that generalize both (1.1), (1.2) and (1.3), (1.4) are introduced. Their mapping properties, product rules, index laws, inverse and composition structures are also studied.

### 2. Definitions and Special Cases

The multidimensional modified fractional integral operators \( S^\alpha,\beta,\eta_+; n \) and \( S^\alpha,\beta,\eta_-; n \) are defined as follows:

\[
S^\alpha,\beta,\eta_+; n f(x) = \frac{1}{\Gamma(\alpha + 1)} \frac{\partial^n}{\partial x_1 \cdots \partial x_n} \int_{\mathbb{R}^n_+} \left[ \min \left\{ \frac{x_1}{t_1}, \ldots, \frac{x_n}{t_n} \right\} - 1 \right]^\alpha \\
\cdot _2F_1 \left( \alpha + \beta, \alpha + \eta; 1 + \alpha; 1 - \min \left\{ \frac{x_1}{t_1}, \ldots, \frac{x_n}{t_n} \right\} \right) f(t) dt,
\]

(2.1)

\[
S^\alpha,\beta,\eta_-; n f(x) = \frac{(-1)^n}{\Gamma(\alpha + 1)} \frac{\partial^n}{\partial x_1 \cdots \partial x_n} \int_{\mathbb{R}^n_+} \left[ 1 - \max \left\{ \frac{x_1}{t_1}, \ldots, \frac{x_n}{t_n} \right\} \right]^\alpha \\
\cdot _2F_1 \left( \alpha + \beta, -\eta; 1 + \alpha; 1 - \max \left\{ \frac{x_1}{t_1}, \ldots, \frac{x_n}{t_n} \right\} \right) f(t) dt
\]

(2.2)

for \( \text{Re}(\alpha) > 0 \).

If \( a = 0 \), then \( _2F_1(a, b; c; z) = 1 \), and therefore, in case \( \beta = -\alpha \) the operators (2.1) and (2.2) reduce to the modified fractional operators (1.1) and (1.2), respectively.
Let now \( n = 1 \). We have
\[
S^{\alpha,\beta,\eta}_{+;1} f(x) = \frac{1}{\Gamma(\alpha + 1)} \frac{d}{dx} \int_0^x \left( \frac{x}{t} - 1 \right)^{\alpha} 2F_1 \left( \alpha + \beta, \alpha + \eta; 1 + \alpha; 1 - \frac{x}{t} \right) f(t) dt.
\]
Using the formula [2]
\[
2F_1(a, b; c; z) = (1 - z)^{-a} 2F_1 \left( a, c - b; c; \frac{z}{z - 1} \right), \quad a, b, c, z \in \mathbb{C},
\]
we obtain
\[
S^{\alpha,\beta,\eta}_{+;1} f(x) = \frac{1}{\Gamma(\alpha + 1)} \frac{d}{dx} \left[ x^{-\alpha-\beta} \int_0^x (x - t)^{\alpha} 2F_1 \left( \alpha + \beta, 1 - \eta; 1 + \alpha; 1 - \frac{t}{x} \right) t^\beta f(t) dt \right]
= \frac{d}{dx} J^{\alpha+1,\beta-1,\eta-1;\beta}_{x^\alpha} f(x).
\]
Then in view of [5, formula (4.1)] we find that
\[
I^{\alpha,\beta,\eta}_x f(x) = S^{\alpha,\beta,\eta}_{+;1} x^{-\beta} f(x),
\]
which means operator (1.3) is a special case of operator (2.1).

Similarly, we have
\[
S^{\alpha,\beta,\eta}_{-;1} f(x) = -\frac{1}{\Gamma(\alpha + 1)} \frac{d}{dx} \int_x^\infty \left( 1 - \frac{x}{t} \right)^{\alpha} 2F_1 \left( \alpha + \beta, -\eta; 1 + \alpha; 1 - \frac{1}{t} \right) f(t) dt
= -\frac{d}{dx} J^{\alpha+1,\beta-1,\eta;\beta}_{x^\alpha} f(x)
\]
and from [5, formula (4.2)] we obtain
\[
J^{\alpha,\beta,\eta}_x f(x) = S^{\alpha,\beta,\eta}_{-;1} x^{-\beta} f(x),
\]
and that means operator (1.4) is a special case of operator (2.2).

By dividing \( \mathbb{R}^n_+ \) for a fixed \( x \in \mathbb{R}^n_+ \) into \( n \) subdomains with zero-measure intersection
\[
\mathbb{R}^n_+ = \bigcup_{k=1}^n \left\{ t \in \mathbb{R}^n_+ \mid \frac{x_k}{t_k} \leq \frac{x_j}{t_j} \ (j = 1, \cdots, n; \ j \neq k) \right\},
\]
the multidimensional fractional operator \( S^{\alpha,\beta,\eta}_{x,n} \) can be expressed as a finite sum of single integrals
\[
S^{\alpha,\beta,\eta}_{x,n} f(x) = \frac{1}{\Gamma(\alpha + 1)} \cdot \sum_{k=1}^n \frac{\partial}{\partial x_k} \left[ x_k \int_0^1 t^{n-\alpha-1} (1 - t)^{\alpha} 2F_1 \left( \alpha + \beta, \alpha + \eta; 1 + \alpha; 1 - \frac{1}{t} \right) f(x_1 t, \cdots, x_n t) dt \right].
\]
Similarly, by dividing $\mathbb{R}^n_+$ into $n$ subdomains with zero-measure intersection

$$
\mathbb{R}^n_+ = \bigcup_{k=1}^n \left\{ t \in \mathbb{R}^n_+ \left| \frac{x_k}{t_k} \geq \frac{x_j}{t_j} \ (j = 1, \cdots, n; \ j \neq k) \right. \right\}
$$

we obtain

$$
S_{-\alpha,\beta,\eta}^{\alpha,\beta,\eta} f(x) = -\frac{1}{\Gamma(\alpha + 1)} \cdot \sum_{k=1}^n \frac{\partial}{\partial x_k} \left[ x_k \int_1^\infty t^{n-\alpha-1} (t - 1)^\alpha \ 2F1 \left( \alpha + \beta, -\eta; 1 + \alpha; 1 - \frac{1}{t} \right) f(x_1t, \cdots, x_nt) dt \right].
$$

3. Fractional Integrals on Space $\mathcal{M}_c \left( \mathbb{R}^n_+ \right)$

Let $f^*(\sigma)$ be the one-dimensional Mellin transform of a function $f(\tau)$ such that $\tau^{\sigma-1} f(\tau) \in L_1(\mathbb{R}_+)$ [7]

$$
\mathfrak{M} \{ f \} (\sigma) = f^*(\sigma) = \int_0^\infty \tau^{\sigma-1} f(\tau) d\tau.
$$

If $\text{Re}(s_j) > 0 \ (j = 1, \cdots, n)$ and $\tau^{s-1} f(\tau) \in L_1(\mathbb{R}_+)$, then it is proved in [8] that the $n$-dimensional Mellin transform of $f(\max[x_1, \cdots, x_n])$ can be evaluated with the help of its one-dimensional Mellin transform as

$$
\int_{\mathbb{R}^n_+} x^{s-1} f(\max[x_1, \cdots, x_n]) dx = \frac{s_1}{s^1} s^1 f^*(s.1),
$$

where $s^1$ means the product $s_1 \cdots s_n$. Similarly, if $\text{Re}(s_j) < 0 \ (j = 1, \cdots, n)$ and $\tau^{s-1} f(\tau) \in L_1(\mathbb{R}_+)$, then [8]

$$
\int_{\mathbb{R}^n_+} x^{s-1} f(\min[x_1, \cdots, x_n]) dx = (-1)^{n-1} \frac{s_1}{s^1} s^1 f^*(s.1).
$$

First we try to find the fractional integrals of the elementary function $x^{-s}$. We have

$$
S_{+\alpha,\beta,\eta}^{\alpha,\beta,\eta} x^{-s} = \frac{1}{\Gamma(\alpha + 1)} \frac{\partial^n}{\partial x_1 \cdots \partial x_n} \int_{\mathbb{R}^n_+} \left[ \min \left\{ \frac{x_1}{t_1}, \cdots, \frac{x_n}{t_n} \right\} - 1 \right]^\alpha
$$

$$
\cdot 2F1 \left( \alpha + \beta, \alpha + \eta; 1 + \alpha; 1 - \text{min} \left\{ \frac{x_1}{t_1}, \cdots, \frac{x_n}{t_n} \right\} \right) t^{-s} dt
$$

$$
= \frac{1}{\Gamma(\alpha + 1)} \frac{\partial^n x^{1-s}}{\partial x_1 \cdots \partial x_n} \int_{\mathbb{R}^n_+} \left[ \min \{ t_1, \cdots, t_n \} - 1 \right]^\alpha
$$

$$
\cdot 2F1 \left( \alpha + \beta, \alpha + \eta; 1 + \alpha; 1 - \text{min} \{ t_1, \cdots, t_n \} \right) t^{-2} dt.
$$
Since the function $2F_1(\alpha + \beta, \alpha + \eta; 1 + \alpha; 1 - t)$ has the asymptotics $O(t^{-\text{Re}(\alpha) - \min[\text{Re}(\beta), \text{Re}(\eta)]})$ at infinity and $O(1)$ at $t = 1$, then $t^{(s-1)\alpha} 2F_1(\alpha + \beta, \alpha + \eta; 1 + \alpha; 1 - t) \in L_1(\mathbb{R}_+)$, provided that $\text{Re}(s.1) < n + \min[\text{Re}(\beta), \text{Re}(\eta)]$ and $\text{Re}(\alpha) > -1$. Hence, if

$$
(3.4) \quad \text{Re}(\alpha) > -1; \quad \text{Re}(s_j) < 1 \quad (j = 1, \ldots, n); \quad \text{Re}(s.1) < n + \min[\text{Re}(\beta), \text{Re}(\eta)],
$$

one can apply formula (3.2) to the last integral in (3.3). Using formula [3, §8.4.49.24]

$$
\mathfrak{M} \left\{ (x - 1)^{c-1} 2F_1(a, b; c; 1 - x) \right\} (s) = \frac{\Gamma(c) \Gamma(1 + a - c - s) \Gamma(1 + b - c - s)}{\Gamma(1 - s) \Gamma(1 + a + b - c - s)}
$$

for $\text{Re}(c) > 0, \text{Re}(s) > 1 + \min[\text{Re}(a - c), \text{Re}(b - c)]$, we get

$$
\int_{\mathbb{R}_+^n} t^{s-2} [\min \{t_1, \ldots, t_n\} - 1]_+^\alpha 2F_1(\alpha + \beta, \alpha + \eta; 1 + \alpha; 1 - \min \{t_1, \ldots, t_n\}) \ dt = (-1)^{n-1} \frac{(s.1 - n) \Gamma(\alpha + 1) \Gamma(\beta + n - s.1) \Gamma(\eta + n - s.1)}{(s - 1)^1 \Gamma(1 + n - s.1) \Gamma(\alpha + \beta + \eta + n - s.1)}
$$

under the conditions (3.4). Thus

$$
(3.5) \quad S_{s.1}^{\alpha, \beta, \eta} x^{-s} = \frac{\Gamma(\beta + n - s.1) \Gamma(\eta + n - s.1)}{\Gamma(n - s.1) \Gamma(\alpha + \beta + \eta + n - |s|)} x^{-s}
$$

provided the conditions (3.4) are satisfied.

Similarly, using formula [3, §8.4.49.22]

$$
\mathfrak{M} \left\{ (1 - x)^{c-1} 2F_1(a, b; c; 1 - x) \right\} (s) = \frac{\Gamma(c) \Gamma(s) \Gamma(s + c - a - b)}{\Gamma(s + c - a) \Gamma(s + c - b)}
$$

for $\text{Re}(c) > 0, \text{Re}(s) > \max[0, \text{Re}(a + b - c)]$ and formula (3.1), we obtain

$$
(3.6) \quad S_{s.1}^{\alpha, \beta, \eta} x^{-s} = \frac{\Gamma(1 - n + s.1) \Gamma(1 - \beta + \eta - n + s.1)}{\Gamma(1 - \beta - n + s.1) \Gamma(1 + \alpha + \eta - n + s.1)} x^{-s}
$$

provided $\text{Re}(\alpha) > -1; \quad \text{Re}(s_j) > 1 \quad (j = 1, \ldots, n); \quad \text{Re}(s.1) > n + \text{Re}(\beta - \eta) - 1$.

Let now $c = (c_1, \ldots, c_n) \in \mathbb{R}^n$ and

$$
\int_{(c) - i\infty}^{(c) + i\infty} f^*(s) ds = \int_{c_1 - i\infty}^{c_1 + i\infty} \cdots \int_{c_n - i\infty}^{c_n + i\infty} f^*(s) ds_1 \cdots ds_n.
$$

The space $\mathfrak{M}_c(\mathbb{R}_+^n)$ is defined in [8] through the set of entire functions of exponential type. It is proved there that $f \in \mathfrak{M}_c(\mathbb{R}_+^n)$ if and only if $f$ can be represented as the inverse Mellin transform

$$
f(x) = \frac{1}{(2\pi i)^n} \int_{(c) - i\infty}^{(c) + i\infty} f^*(s) x^{-s} ds
$$

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of a function $f^*(s)$ infinitely differentiable and with compact support on $((c) - i\infty, (c) + i\infty)$.

**Theorem 1. a)** Let $\Re(\alpha) > 0$; $c_j + \Re(d_j) < 1$ $(j = 1, \ldots, n)$; $\alpha + \beta + \eta + n - d.1 - c.1 \neq 0, -1, \cdots; c.1 + \Re(d.1) < n + \min[\Re(\beta), \Re(\eta)]$. Then $x^d S^{\alpha, \beta, \eta}_{+n} x^{-d}$ is a homeomorphism of the space $\mathcal{M}_c \left( \mathbb{R}^n_+ \right)$ onto itself. Moreover, it can be written in the form

$$\Gamma(\beta + n - d.1 - s.1) \Gamma(\eta + n - d.1 - s.1) \ f^*(s)x^{-s}ds. \tag{3.7}$$

**b)** Let $\Re(\alpha) > 0$; $c_j + \Re(d_j) > 1$ $(j = 1, \ldots, n)$; $-\beta - n + d.1 + c.1 \neq -1, -2, \cdots; \alpha + \eta - n + d.1 + c.1 \neq -1, -2, \cdots; c.1 + \Re(d.1) > n + \Re(\beta - \eta) - 1$. Then $x^d S^{\alpha, \beta, \eta}_{-n} x^{-d}$ is a homeomorphism of the space $\mathcal{M}_c \left( \mathbb{R}^n_+ \right)$ onto itself, moreover,

$$\Gamma(1 - n + d.1 + s.1) \Gamma(1 - \beta + \eta - n + d.1 + s.1) \ f^*(s)x^{-s}ds. \tag{3.8}$$

**Proof.** We consider now the operator $x^d S^{\alpha, \beta, \eta}_{+n} x^{-d}$. The proof for the operator $x^d S^{\alpha, \beta, \eta}_{-n} x^{-d}$ follows in a similar manner. Since $f \in \mathcal{M}_c \left( \mathbb{R}^n_+ \right)$ we have

$$x^d S^{\alpha, \beta, \eta}_{-n} x^{-d} f(x) = x^d S^{\alpha, \beta, \eta}_{-n} x^{-d} \left( \frac{1}{(2\pi i)^n} \int_{(c) - i\infty}^{(c) + i\infty} f^*(s)x^{-s}ds \right). \tag{3.9}$$

The interchange of the order of integration is possible, since $f^*(s)$ has a compact support. Using now formula (3.6), we obtain (3.8). Function

$$\frac{\Gamma(1 - n + d.1 + s.1) \Gamma(1 - \beta + \eta - n + d.1 + s.1)}{\Gamma(1 - \beta - n + d.1 + s.1) \Gamma(1 + \alpha + \eta - n + d.1 + s.1)} f^*(s)$$

has a compact support and is infinitely differentiable on $((c) - i\infty, (c) + i\infty)$ if and only if so does $f^*(s)$. Hence $x^d S^{\alpha, \beta, \eta}_{-n} x^{-d}$ is a bijection on $\mathcal{M}_c \left( \mathbb{R}^n_+ \right)$. The continuity of the mapping $f \to x^d S^{\alpha, \beta, \eta}_{-n} x^{-d}$ in $\mathcal{M}_c \left( \mathbb{R}^n_+ \right)$ is also obvious.
Let $d = 0$ in Theorem 1, then we obtain

**Corollary. a)** Let $\text{Re}(\alpha) > 0$; $c_j < 1$ $(j = 1, \ldots, n)$; $\alpha + \beta + \eta + n - c.1 \neq 0, -1, \cdots$; $c.1 < n + \min \{\text{Re}(\beta), \text{Re}(\eta)\}$. Then $S_{\alpha,\beta,\eta}^{\alpha,\beta,\eta}$ is a homeomorphism of the space $M_n^c \left( \mathbb{R}^n_+ \right)$ onto itself and

$$
S_{\alpha,\beta,\eta}^{\alpha,\beta,\eta} f(x) = \frac{1}{(2\pi i)^n} \int_{(c)-i\infty}^{(c)+i\infty} \frac{\Gamma(\beta + n - s.1)\Gamma(\eta + n - s.1)}{\Gamma(n - s.1)\Gamma(\alpha + \beta + \eta + n - s.1)} f^*(s)x^{-s}ds.
$$

**b)** Let $\text{Re}(\alpha) > 0$; $c_j > 1$ $(j = 1, \ldots, n)$; $-\beta - n + c.1 \neq -1, -2, \cdots$; $\alpha + \eta - n + c.1 \neq -1, -2, \cdots$; $c.1 > n + \text{Re}(\beta - \eta) - 1$. Then the operator $S_{\alpha,\beta,\eta}^{\alpha,\beta,\eta}$ is a homeomorphism of the space $M_n^c \left( \mathbb{R}^n_+ \right)$ onto itself and

$$
S_{\alpha,\beta,\eta}^{\alpha,\beta,\eta} f(x) = \frac{1}{(2\pi i)^n} \int_{(c)-i\infty}^{(c)+i\infty} \frac{\Gamma(1 - n + s.1)\Gamma(1 - \beta + \eta - n + s.1)}{\Gamma(1 - \beta - n + s.1)\Gamma(1 + \alpha + \eta - n + s.1)} f^*(s)x^{-s}ds.
$$

4. Fractional Integrals in $L_2$-spaces

Let the condition of Theorem 1, a) be satisfied. Let $f \in M_n^c \left( \mathbb{R}^n_+ \right)$. Since $f^*(s)$ has a compact support and is continuous on $\left((c) - i\infty, (c) + i\infty\right)$, then $f^*(s) \in L_2 \left((c) - i\infty, (c) + i\infty\right)$. Hence, by the Plancherel theorem for the Mellin transform [1] $x^{c-1/2}f(x) \in L_2 \left( \mathbb{R}^n_+ \right)$ and

$$
\|x^{c-1/2}f(x)\|_{L_2(\mathbb{R}^n_+)} = \frac{1}{(2\pi)^n} \|f^*(s)\|_{L_2((c)-i\infty,(c)+i\infty)}.
$$

Using the Plancherel theorem now for the function $x^dS_{\alpha,\beta,\eta}^{\alpha,\beta,\eta} x^{-d}f(x)$ instead of $f(x)$, and remembering that its Mellin transform is

$$
\mathcal{M} \left\{x^dS_{\alpha,\beta,\eta}^{\alpha,\beta,\eta} x^{-d}f(x)\right\}(s) = \frac{\Gamma(\beta + n - d.1 - s.1)\Gamma(\eta + n - d.1 - s.1)}{\Gamma(n - d.1 - s.1)\Gamma(\alpha + \beta + \eta + n - d.1 - s.1)} f^*(s),
$$

(see formula (3.7)) we obtain

$$
\|x^{c+d-1/2}S_{\alpha,\beta,\eta}^{\alpha,\beta,\eta} x^{-d}f(x)\|_{L_2(\mathbb{R}^n_+)} = \frac{1}{(2\pi)^n} \left\| \frac{\Gamma(\beta + n - d.1 - s.1)\Gamma(\eta + n - d.1 - s.1)}{\Gamma(n - d.1 - s.1)\Gamma(\alpha + \beta + \eta + n - d.1 - s.1)} f^*(s) \right\|_{L_2((c)-i\infty,(c)+i\infty)}.
$$
From the Stirling formula for the Gamma function [2] we see that (4.2) is uniformly bounded with respect to \(s \in ((c) - i\infty, (c) + i\infty)\), or more precisely, that it decays like \(|s|^{-\alpha}\) as \(|s| \to \infty\) on \(((c) - i\infty, (c) + i\infty)\). Hence

\[
\left\| \frac{\Gamma(\beta + n - d.1 - s.1)\Gamma(\eta + n - d.1 - s.1)}{\Gamma(n - d.1 - s.1)\Gamma(\alpha + \beta + \eta + n - d.1 - s.1)} f^*(s) \right\|_{L_2((c) - i\infty, (c) + i\infty)} \leq M \| f^*(s) \|_{L_2((c) - i\infty, (c) + i\infty)}.
\]

Using now formula (4.1) we get

\[
(4.3) \quad \left\| x^{c+d-1/2} s_{a_{\alpha},b_{\beta},c_{\gamma}} x^{-d} f(x) \right\|_{L_2(\mathbb{R}^n_+)} \leq M \left\| x^{c-1/2} f(x) \right\|_{L_2(\mathbb{R}^n_+)}.
\]

Similarly, if the condition of Theorem 1, b) is satisfied and \(f \in M_c \left( \mathbb{R}^n_+ \right)\), then

\[
(4.4) \quad \left\| x^{c+d-1/2} s_{a_{\alpha},b_{\beta},c_{\gamma}} x^{-d} f(x) \right\|_{L_2(\mathbb{R}^n_+)} \leq M \left\| x^{c-1/2} f(x) \right\|_{L_2(\mathbb{R}^n_+)}.
\]

Since the set of compactly supported and infinitely differentiable functions on \((c) - i\infty, (c) + i\infty\) is dense in \(L_2 \left( (c) - i\infty, (c) + i\infty \right)\) in \(L_2\)-topology, from the Plancherel theorem for the Mellin transform we see that the space \(M_c \left( \mathbb{R}^n_+ \right)\) is dense in the space \(L_2 \left( \mathbb{R}^n_+; x^{c-1/2} \right)\) of functions \(f : \mathbb{R}^n_+ \to \mathbb{C}\) such that

\[
\|f\|_{L_2(\mathbb{R}^n_+; x^{c-1/2})} \equiv \left\{ \int_{\mathbb{R}^n_+} \left| x_1^{c_1-1/2} \cdots x_n^{c_n-1/2} f(x) \right|^2 \, dx \right\}^{1/2} < \infty.
\]

Thus from (4.3) and (4.4) we find that \(x^d s_{a_{\alpha},b_{\beta},c_{\gamma}} x^{-d}\) and \(x^d s_{a_{\alpha},b_{\beta},c_{\gamma}} x^{-d}\) can be extended from \(M_c \left( \mathbb{R}^n_+ \right)\) into bounded operators in \(L_2 \left( \mathbb{R}^n_+; x^{c-1/2} \right)\). It is not difficult to prove that, in this case, the extended operators \(s_{a_{\alpha},b_{\beta},c_{\gamma}} \) and \(s_{a_{\alpha},b_{\beta},c_{\gamma}} \) have the same forms as (2.1) and (2.2). We have just obtained

**Theorem 2.** Let conditions of Theorem 1, a) be satisfied for the operator \(x^d s_{a_{\alpha},b_{\beta},c_{\gamma}} x^{-d}\), and of Theorem 1, b) for \(x^d s_{a_{\alpha},b_{\beta},c_{\gamma}} x^{-d}\). Then the operators \(x^d s_{a_{\alpha},b_{\beta},c_{\gamma}} x^{-d}\) and \(x^d s_{a_{\alpha},b_{\beta},c_{\gamma}} x^{-d}\) are bounded in \(L_2 \left( \mathbb{R}^n_+; x^{c-1/2} \right)\).

5. Integration by Parts

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The following formula connects operators $S_{+;n}^{\alpha,\beta,n}$ and $S_{-;n}^{\alpha,\beta,n}$ in the manner similar to the relation between the Riemann-Liouville and the Weyl fractional integral operators.

**Theorem 3.** Let $f \in \mathcal{M}_c \left( \mathbb{R}_+^n \right)$ and $g \in \mathcal{M}_{1-\text{Re}(d)-c} \left( \mathbb{R}_+^n \right)$ and $d.1 = 1 - n - \beta$. Then
\begin{equation}
(5.1) \quad \int_{\mathbb{R}_+^n} x^{-d} g(x) S_{+;n}^{\alpha,\beta,n} f(x) dx = \int_{\mathbb{R}_+^n} x^{-d} f(x) S_{-;n}^{\alpha,\beta,n} g(x) dx,
\end{equation}
provided that $\text{Re}(\alpha) > 0; c_j < 1, c_j + \text{Re}(d_j) < 0, (j = 1, \cdots, n); c.1 < n + \min[\text{Re}(\beta), \text{Re}(\eta)]; c.1 + \text{Re}(d.1) < 1 - \text{Re}(\beta - \eta)$.

**Proof.** Under the conditions of Theorem 3 the operator $S_{+;n}^{\alpha,\beta,n}$ is a homeomorphism on $\mathcal{M}_c \left( \mathbb{R}_+^n \right)$ and $S_{-;n}^{\alpha,\beta,n}$ on $\mathcal{M}_{1-\text{Re}(d)-c} \left( \mathbb{R}_+^n \right)$. We have
\begin{align*}
\int_{\mathbb{R}_+^n} x^{-d} g(x) S_{+;n}^{\alpha,\beta,n} f(x) dx &= \int_{\mathbb{R}_+^n} x^{-d} g(x) \frac{1}{(2\pi i)^n} \int_{(c)-i\infty}^{(c)+i\infty} \frac{\Gamma(\beta + n - s.1)\Gamma(\eta + n - s.1)}{\Gamma(n - s.1)\Gamma(\alpha + \beta + \eta + n - s.1)} f^*(s)x^{-s} ds dx \\
&= \frac{1}{(2\pi i)^n} \int_{(c)-i\infty}^{(c)+i\infty} \frac{\Gamma(\beta + n - s.1)\Gamma(\eta + n - s.1)}{\Gamma(n - s.1)\Gamma(\alpha + \beta + \eta + n - s.1)} f^*(s) \int_{\mathbb{R}_+^n} x^{-d-s} g(x) dx ds \\
&= \frac{1}{(2\pi i)^n} \int_{(c)-i\infty}^{(c)+i\infty} \frac{\Gamma(\beta + n - s.1)\Gamma(\eta + n - s.1)}{\Gamma(n - s.1)\Gamma(\alpha + \beta + \eta + n - s.1)} f^*(s)g^*(1 - d - s) ds.
\end{align*}
By changing $s$ by $1 - d - s$ we get
\begin{equation}
(5.2) \quad \int_{\mathbb{R}_+^n} x^{-d} g(x) S_{+;n}^{\alpha,\beta,n} f(x) dx
\end{equation}
\begin{align*}
&= \frac{1}{(2\pi i)^n} \int_{(1-\text{Re}(d)-c)-i\infty}^{(1-\text{Re}(d)-c)+i\infty} \frac{\Gamma(\beta + d.1 + s.1)\Gamma(\eta + d.1 + s.1)}{\Gamma(d.1 + s.1)\Gamma(\alpha + \beta + \eta + d.1 + s.1)} g^*(s) \int_{\mathbb{R}_+^n} x^{-d-s} f(x) dx ds \\
&= \int_{\mathbb{R}_+^n} x^{-d} f(x) \frac{1}{(2\pi i)^n} \int_{(1-\text{Re}(d)-c)-i\infty}^{(1-\text{Re}(d)-c)+i\infty} \frac{\Gamma(\beta + d.1 + s.1)\Gamma(\eta + d.1 + s.1)}{\Gamma(d.1 + s.1)\Gamma(\alpha + \beta + \eta + d.1 + s.1)} x^{-s} g^*(s) ds dx.
\end{align*}
Using the assumption $d.1 = 1 - n - \beta$, the last inner integral becomes
\begin{align*}
&\frac{1}{(2\pi i)^n} \int_{(1-\text{Re}(d)-c)-i\infty}^{(1-\text{Re}(d)-c)+i\infty} \frac{\Gamma(1 - n + s.1)\Gamma(1 - \beta + \eta - n + s.1)}{\Gamma(1 - \beta - n + s.1)\Gamma(1 + \alpha + \eta - n + s.1)} g^*(s)x^{-s} ds. \\
&= S_{-;n}^{\alpha,\beta,n} g(x).
\end{align*}
Hence, formula (5.1) is proved.

6. Transformation and Index Laws

In view of the Mellin inversion type expressions (3.10) and (3.11) of the operators $S_{+;n}^{\alpha,\beta,\eta}$ and $S_{-;n}^{\alpha,\beta,\eta}$, we can easily find out that under the same assumptions as Corollary of Theorem 1 there hold the formulas

\[(6.1) \quad S_{+;n}^{\alpha,\beta,\eta} f(x) = S_{+;n}^{\alpha,\beta} f(x), \]
\[(6.2) \quad S_{-;n}^{\alpha,\beta,\eta} f(x) = S_{-;n}^{\alpha-\eta,\alpha-\beta} f(x). \]

Such formulas can be obtained from Theorem 1 for each of the operators as follows:

**Theorem 4.** a) Let $\text{Re}(\alpha) > 0$; $c_j < 1 - \max[0, \text{Re}(g_j)]$ (j = 1, ⋯, n) with g.1 = \(-\alpha - \beta - \eta\); c.1 < n + min[Re(\beta), Re(\eta)]. Then in $\mathcal{M}_c(\mathbb{R}_+^n)$ there holds the relation

\[(6.3) \quad S_{+;n}^{\alpha,\beta,\eta} f(x) = x^g S_{+;n}^{\alpha-\eta,\alpha-\beta} x^{-g} f(x). \]

b) Let $\text{Re}(\alpha) > 0$; $c_j > 1 - \min[0, \text{Re}(g_j)]$ (j = 1, ⋯, n) with g.1 = \(-\beta + \eta\); \(-\beta - n + c.1 \neq -1, -2, \cdots; \alpha + \eta - n + c.1 \neq -1, -2, \cdots\); Then in $\mathcal{M}_c(\mathbb{R}_+^n)$ there holds the relation

\[(6.4) \quad S_{-;n}^{\alpha,\beta,\eta} f(x) = x^g S_{-;n}^{\alpha-\eta,\beta} x^{-g} f(x). \]

**Theorem 5.** Let $\text{Re}(\alpha) > -\text{Re}(\beta) > 0$ and $f \in \mathcal{M}_c(\mathbb{R}_+^n)$.

a) If $c_j + \max[0, \text{Re}(d_j)] < 1$ (j = 1, ⋯, n); n - c.1 \neq 0, -1, ⋯; c.1 < n + \text{Re}(\beta)$; c.1 < n - \text{Re}(\alpha + \beta) - \text{Re}(d.1), then

\[(6.5) \quad S_{+;n}^{\alpha,\beta-\alpha-\beta-d.1} f(x) = X_{+;n}^{\alpha-\beta} x^d X_{+;n}^{\alpha+\beta} x^{-d} f(x) = x^d X_{+;n}^{\alpha+\beta} x^{-d} X_{+;n}^{\alpha-\beta} f(x). \]

b) If $c_j + \max[0, \text{Re}(d_j)] < 1$ (j = 1, ⋯, n); n - c.1 \neq 0, -1, ⋯; c.1 < n - \max[\text{Re}(\alpha + \beta), \text{Re}(d.1 - \beta)]$, then

\[(6.6) \quad S_{+;n}^{\alpha-\beta,\beta-\alpha-\beta-d.1} f(x) = X_{+;n}^{\alpha+\beta} x^d X_{+;n}^{\alpha-\beta} x^{-d} f(x) = x^d X_{+;n}^{\alpha-\beta} x^{-d} X_{+;n}^{\alpha+\beta} f(x). \]
c) If \(c_j + \min[0, \Re(d_j)] > 1 \quad (j = 1, \ldots, n); \quad -\beta - n + c.1 \neq -1, -2, \ldots; \quad \beta + d.1 - n + c.1 \neq -1, -2, \ldots; \quad c.1 > n - 1 - \min[0, \Re(d.1)], \) then

\[
S_{-n}^{\alpha, \beta + d.1} f(x) = X_{-n}^{-\beta} x^d X_{-n}^{\alpha + \beta} x^{-d} f(x) = x^d X_{-n}^{\alpha + \beta} x^{-d} X_{-n}^{-\beta} f(x).
\]

(6.7)

\[
S_{-n}^{\alpha, \beta + d.1} f(x) = X_{-n}^{-\beta} x^d X_{-n}^{\alpha + \beta} x^{-d} f(x) = x^d X_{-n}^{\alpha + \beta} x^{-d} X_{-n}^{-\beta} f(x).
\]

d) If \(c_j + \min[0, \Re(d_j)] > 1 \quad (j = 1, \ldots, n); \quad -\beta - n + d.1 + c.1 \neq -1, -2, \ldots; \quad \alpha + \beta - n + c.1 \neq -1, -2, \ldots; \quad c.1 > n - 1 - \min[0, \Re(d.1)], \) then

(6.8)

\[
S_{-n}^{\alpha, \beta + d.1} f(x) = X_{-n}^{-\beta} x^d X_{-n}^{\alpha + \beta} x^{-d} f(x) = x^d X_{-n}^{\alpha + \beta} x^{-d} X_{-n}^{-\beta} f(x).
\]

**Proof.** We prove only the first part of (6.5). Under the corresponding conditions, all operators \(S_{-n}^{\alpha, \beta - \alpha - \beta - d.1}, X_{-n}^{-\beta}, X_{-n}^{\alpha + \beta}\) are homeomorphism in \(\mathcal{M}_c \left( \mathbb{R}_+^n \right) \). We have, by referring to formula \([8, (5.2)]\) for \(f \in \mathcal{M}_c \left( \mathbb{R}_+^n \right)\),

\[
X_{-n}^{-\beta} x^d X_{-n}^{\alpha + \beta} x^{-d} f(x)
= X_{-n}^{-\beta} \left( \frac{1}{(2\pi i)^n} \int_{(c)-i\infty}^{(c)+i\infty} \frac{\Gamma(n - \alpha - \beta - d.1 - s.1)}{\Gamma(n - d.1 - s.1)} f^*(s)x^{-s} ds \right)
= \frac{1}{(2\pi i)^n} \int_{(c)-i\infty}^{(c)+i\infty} \frac{\Gamma(n + \beta - s.1) \Gamma(n - \alpha - \beta - d.1 - s.1)}{\Gamma(n - s.1) \Gamma(n - d.1 - s.1)} f^*(s)x^{-s} ds
= S_{-n}^{\alpha, \beta, \beta + d.1} f(x).
\]

The other formulas are proved by similar ways using formulas \([8, (5.2), (5.5)]\).

**Theorem 6.** Let \(\Re(\alpha) > 0, \Re(\gamma) > 0\) and \(f \in \mathcal{M}_c \left( \mathbb{R}_+^n \right)\).

a) If \(c_j + \max[0, \Re(d_j)] < 1 \quad (j = 1, \ldots, n); \quad \alpha + \beta - \gamma - d.1 + n - c.1 \neq 0, -1, \ldots; \quad c.1 < n + \min[\Re(\beta), \Re(-\gamma - d.1)], \) then

(6.9)

\[
S_{+n}^{\alpha + \gamma, \beta - \gamma - d.1} f(x) = x^d X_{+n}^{\gamma} x^{-d} S_{+n}^{\alpha, \beta - d.1} f(x) = S_{+n}^{\alpha, \beta, -d.1} x^d X_{+n}^{\gamma} x^{-d} f(x).
\]

b) If \(c_j + \max[0, \Re(d_j)] < 1 \quad (j = 1, \ldots, n); \quad -d.1 + n - c.1 \neq 0, -1, \ldots; \quad c.1 < n + \min[\Re(\beta), \Re(-\alpha - \beta - \gamma - d.1)], \) then

(6.10)

\[
S_{+n}^{\alpha + \gamma, \beta - \alpha - \gamma - d.1} f(x) = x^d X_{+n}^{\gamma} x^{-d} S_{+n}^{\alpha, \beta - \gamma - d.1} f(x)
= S_{+n}^{\alpha, \beta, -\alpha - \beta - \gamma - d.1} x^d X_{+n}^{\gamma} x^{-d} f(x).
\]

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c) If \( c_j + \min[0, \Re(d_j)] > 1 \) \((j = 1, \cdots, n)\); \(-\beta - n + c.1 \neq -1, -2, \cdots; \gamma + d.1 - n + c.1 \neq -1, -2, \cdots; c.1 > n + \Re(\alpha + \beta - d.1) - 1\), then

\[
(6.11) \quad S_{\gamma-\alpha}^{\alpha+\gamma, \beta, -\alpha+d.1} f(x) = x^d X_{\gamma-\alpha}^{\alpha+\gamma, \beta} x^{-d} S_{\gamma-\alpha}^{\alpha+\gamma, \beta, -\alpha+d.1} f(x) = S_{\gamma-\alpha}^{\alpha+\gamma, \beta, -\alpha+d.1} x^d X_{\gamma-\alpha}^{\alpha+\gamma, \beta} x^{-d} f(x).
\]

\[
(6.12) \quad S_{\gamma-\alpha}^{\alpha+\gamma, \beta, \beta+d.1} f(x) = x^d X_{\gamma-\alpha}^{\alpha+\gamma, \beta} x^{-d} S_{\gamma-\alpha}^{\alpha+\gamma, \beta, \beta+d.1} f(x) = S_{\gamma-\alpha}^{\alpha+\gamma, \beta, \beta+d.1} x^d X_{\gamma-\alpha}^{\alpha+\gamma, \beta} x^{-d} f(x).
\]

The proof of Theorem 6 is similar to that of Theorem 5. Using Theorem 4 one can get more other similar formulas.

7. Modified Fractional Derivatives

Since \( S_{\gamma-\alpha}^{\alpha+\gamma, \beta, \eta} \) is a homeomorphism of \( \mathfrak{M} \left( \mathbb{R}^n_+ \right) \) onto itself if \( \Re(\alpha) > 0; \Re(c_j) < 1 \) \((j = 1, \cdots, n)\); \( \alpha + \beta + \eta - c.1 \neq -n, -n - 1, \cdots; \Re(c.1) < n + \min[\Re(\beta), \Re(\eta)] \), then there exists its inverse operator which we will define as \( \left( S_{\gamma-\alpha}^{\alpha+\gamma, \beta, \eta} \right)^{-1} \). This operator is also a homeomorphism of \( \mathfrak{M} \left( \mathbb{R}^n_+ \right) \) onto itself. Suppose now the condition of part a) of Theorem 5 is satisfied. Then formula (6.3) is valid. Hence, if we remember that the inverse of \( X_{\gamma-\alpha}^{\alpha+\gamma, \beta, \eta} \) is \( X_{\gamma-\alpha}^{\alpha+\gamma, \beta, \eta} \) [8], we obtain

\[
(7.1) \quad \left( S_{\gamma-\alpha}^{\alpha+\gamma, \beta, \eta} \right)^{-1} = X_{\gamma-\alpha}^{\alpha+\gamma, \beta, \eta} x^d X_{\gamma-\alpha}^{\alpha+\gamma, \beta, \eta} x^{-d}
\]

with \( d.1 = -\alpha - \beta - \eta \). Let \( k \) be an integer, \( k > \Re(\alpha) \). Then using Theorem 6 from [8]

\[
x^b X_{\gamma-\alpha}^{\alpha+\gamma, \beta} x^{-b} X_{\gamma-\alpha}^{\alpha+\gamma, \beta} = X_{\gamma-\alpha}^{\alpha+\gamma, \beta + k}
\]

with \( b.1 = \beta \) proved there for \( \Re(\beta) > 0 \), but still valid for other value of \( \Re(\beta) \), if \( X_{\gamma-\alpha}^{\alpha+\gamma, \beta} \) is considered as the modified fractional differential operator in \( \mathfrak{M} \left( \mathbb{R}^n_+ \right) \) when \( \Re(\beta) \leq 0 \) [8], we obtain

\[
(7.2) \quad x^b X_{\gamma-\alpha}^{\alpha+\gamma, \beta} x^{-b} \left( S_{\gamma-\alpha}^{\alpha+\gamma, \beta, \eta} \right)^{-1} = X_{\gamma-\alpha}^{\alpha+\gamma, \beta, \eta} x^d X_{\gamma-\alpha}^{\alpha+\gamma, \beta, \eta} x^{-d}
\]

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with $b.1 = \beta$ and $d.1 = -\alpha - \beta - \eta$. Since $k > \text{Re}(\alpha)$, the right hand side of (7.2) is $S^{k-\alpha, -\beta - k, 2\alpha + 2\beta + \eta}_{+;n}$ by (6.5). Hence

$$\text{(7.3)} \quad (S^{\alpha, \beta, \eta}_{+;n})^{-1} = (x^b X^{k-b}_{+;n} x^{-b}) S^{k-\alpha, -\beta - k, 2\alpha + 2\beta + \eta}_{+;n}.$$ 

Using formula (7.5) from [8] for $X^{k-b}_{+;n}$ we obtain formula for the inverse of the fractional integral

$$\text{(7.4)} \quad (S^{\alpha, \beta, \eta}_{+;n})^{-1} f(x) = x^b \prod_{j=1}^{k} \left( n - j + x_1 \frac{\partial}{\partial x_1} + \cdots + x_n \frac{\partial}{\partial x_n} \right) \left[ x^{-b} S^{k-\alpha, -\beta - k, 2\alpha + 2\beta + \eta}_{+;n} \right] f(x)$$

if $c_j < 1$ $(j = 1, \cdots, n)$; $b.1 = \beta \alpha + \beta + \eta + n - c.1 \neq 0, -1, \cdots; c.1 < n + \min[\text{Re}(\beta), \text{Re}(\eta), \text{Re}(-\beta - k), \text{Re}(2\alpha + 2\beta + \eta)].$

Similarly, the inverse for the operator $S^{\alpha, \beta, \eta}_{-;n}$ can be expressed in the way

$$\text{(7.5)} \quad (S^{\alpha, \beta, \eta}_{-;n})^{-1} f(x) = x^b (-1)^k \prod_{j=1}^{k} \left( n - j + x_1 \frac{\partial}{\partial x_1} + \cdots + x_n \frac{\partial}{\partial x_n} \right) \left[ x^{-b} S^{k-\alpha, -\beta - k, -2\beta + \eta - k}_{-;n} \right] f(x)$$

with $b.1 = \beta$ and $k$ being any integer such that $k > \text{Re}(\alpha)$, if $c_j > 1$ $(j = 1, \cdots, n)$; $\beta + k - n + c.1 \neq -1, -2, \cdots; -\alpha - 2\beta + \eta - n + c.1 \neq -1, -2, \cdots; c.1 > n + \text{Re}(\beta - \eta) - 1$.

References


