ON THE RANGE OF THE HANKEL AND EXTENDED HANKEL TRANSFORMS

VU KIM TUAN

Abstract

The range of the Hankel and extended Hankel transforms on some spaces of functions is described. The Paley-Wiener theorem for the Hankel transform is also obtained.

1. INTRODUCTION

The Hankel transform is defined by [27]

\[ f(x) = \langle H_\nu g \rangle(x) = \int_0^\infty \sqrt{xy}J_\nu(xy)g(y) \, dy, \quad x \in \mathbb{R}^+ = (0, \infty), \tag{1} \]

if the integral converges in some sense (absolutely, improper, or mean convergence), where \( J_\nu(x) \) is the Bessel function of the first kind [1]. It is well-known [27] that if \( \Re \nu > -1 \), then the Hankel transform is an automorphism of \( L_2(\mathbb{R}^+) \) and its inverse on \( L_2(\mathbb{R}^+) \) has the symmetric form

\[ g(x) = \langle H_\nu f \rangle(x) = \int_0^\infty \sqrt{xy}J_\nu(xy)f(y) \, dy, \quad x \in \mathbb{R}^+, \ \Re \nu > -1. \tag{2} \]

The Hankel transform has been extended to \( \Re \nu < -1, \ \Re \nu \neq -3, -5, \ldots \), as follows [13, 22]

\[ f(x) = \langle H_\nu g \rangle(x) = \int_0^\infty \sqrt{xy}J_{\nu,m}(xy)g(y) \, dy, \]
\[ x \in \mathbb{R}^+, \ 1 - 2m > \Re \nu > -2m - 1, \ m > 0, \tag{3} \]

where \( J_{\nu,m}(x) \) is the truncated (or "cut") Bessel function of the first kind

\[ J_{\nu,m}(x) = J_\nu(x) - \sum_{k=0}^{m-1} \frac{(-1)^k x^{\nu+2k}}{\Gamma(\nu + k + 1)k!}, \ 1 - 2m > \Re \nu > -2m - 1, \ m \geq 0, \tag{4} \]

and the integral is understood in \( L_2 \) sense. The extended Hankel transform (3) is a bounded operator in \( L_2(\mathbb{R}^+) \) and its inverse, also a bounded operator in \( L_2(\mathbb{R}^+) \), has been proved to have the form [12]

\[ g(x) = -x^{\nu-1/2} \frac{d}{dx}x^{1/2-\nu} \int_0^\infty \sqrt{xy}J_{\nu-1,m+1}(xy)f(y) \, dy, \]
\[ x \in \mathbb{R}^+, \ 1 - 2m > \Re \nu > -2m - 1, \ m > 0. \tag{5} \]

\[ ^1{} 1991 \text{ Mathematics Subject Classification.} \text{ Primary 44} \]

Key words and phrases. Hankel Transform, Range of Integral Transforms, Paley-Wiener Theorem

Supported by the Kuwait University research grant SM 112
Formula (5) can be rewritten in the equivalent form, symmetric to formula (3). Indeed, if we put
\[ f_N(x) = \begin{cases} f(x), & x \in [1/N, N] \\ 0, & \text{otherwise} \end{cases} \tag{6} \]
then \( f_N(x) \) tends to \( f(x) \) in \( L_2(R_+) \) norm. Therefore, if \( g_N(x) \) is the inverse of the extended Hankel transform (5) of \( f_N(x) \), then \( g_N(x) \) tends to \( g(x) \) in \( L_2(R_+) \) norm. Using the relation
\[ \frac{d}{dx} \left( x^{1-\nu} J_{\nu-1,m+1}(x) \right) = -x^{\nu} J_{\nu,m}(x), \quad \Re \nu > -2m - 1, \quad m \geq 0, \tag{7} \]
we have
\[
g_N(x) = -x^{\nu-1/2} \frac{d}{dx} x^{1/2-\nu} \int_{1/N}^N \sqrt{xy} J_{\nu-1,m+1}(xy) f(y) \, dy \\
= \int_{1/N}^N \sqrt{xy} J_{\nu,m}(xy) f(y) \, dy. \tag{8} \]
Therefore,
\[
g(x) = (H_{\nu} f)(x) = \int_0^\infty \sqrt{xy} J_{\nu,m}(xy) f(y) \, dy, \quad 1 - 2m > \Re \nu > -2m - 1, \quad m > 0, \tag{9} \]
where the integral is understood in \( L_2 \) sense.

In [11, 21, 24] the range of the Hankel transform in the space \( L_p \) with weight has been described through the range of the fractional integral operator and the Fourier cosine transform, or through some Parseval relation. In [28, 29] the Hankel transform is proved to be an automorphism of the space of functions \( M_{\gamma}^{-1}(L) \) introduced there. In [14] the range of the Hankel transform of infinitely differentiable functions with compact supports has been discussed. Zemanian [31] has constructed a testing function space \( H_\mu, \mu \in R \), consisting of smooth functions \( \varphi \) on \((0, \infty)\) such that
\[
\gamma_{m,k}^\mu(\varphi) = \sup_{0 < \alpha < \infty} \left| x^m \left( \frac{d}{dx} \right)^k (x^{-\mu-1/2} \varphi(x)) \right| < \infty, \tag{10} \]
and proved that the Hankel transform is an automorphism on \( H_\mu \) if \( \mu > -1/2 \). This space has been generalized by different ways in [3, 4, 5, 6, 7, 8, 9, 10, 15, 16, 17, 18, 19] to deal with the Hankel transform of distributions. All functions of these spaces are smooth and with special conditions on the behaviour at both 0 and infinity.

In this paper we describe the range of the Hankel transform on some spaces of functions. Square integrable functions considered in Theorem 1 are not required to be smooth, and only some conditions at infinity are supposed. The range of the Hankel transform of compactly supported functions which are either square integrable (Paley-Wiener Theorem 2) or infinitely differentiable (Paley-Wiener-Schwartz Theorem 4) is also characterized.
Even in the case of infinitely differentiable functions with compact supports our Theorem 4 is different from [14]. The Hankel (1) and the extended Hankel (3) transforms in some other spaces of functions (analytic functions, space of functions including the space $\mathcal{M}_{c,\gamma}^{-1}(L)$ [28, 29] as a special case) are also considered.

One of the main tools in the proofs of our next two theorems is the Plancherel’s theorem for the Hankel transform

$$\|\mathcal{H}_\nu g\|_2 = \|g\|_2,$$  \hspace{1cm} (11)

where $\|g\|_p = \|g\|_{L_p(R_+)}$, $1 \leq p \leq \infty$, that is valid only when $\nu$ is a real number and $\nu > -1$ ([27]). For complex $\nu$ the Plancherel’s equation (11) is replaced by the inequalities

$$C^{-1}\|g\|_2 \leq \|\mathcal{H}_\nu g\|_2 \leq C\|g\|_2, \quad \Re\nu > -1,$$  \hspace{1cm} (12)

where $C \in [1, \infty)$ is a constant independent of $g$. The inequalities (12) also hold for the extended Hankel transform (3) for $\Re\nu < -1, \Re\nu \neq -3, -5, \ldots$ [22].

2. HANKEL TRANSFORM OF RAPID DECREASING FUNCTIONS

The range of the Hankel transform of rapid decreasing and square integrable functions is described by the following

Theorem 1. Let $y^n g(y) \in L_2(R_+)$ for all $n = 0, 1, 2, \ldots$. A function $f(x)$ is the Hankel transform $\mathcal{H}_\nu$, $\Re\nu \geq 1/2$, of the function $g(y)$ if and only if

i) $f(x)$ is infinitely differentiable on $R_+$;

ii) $\frac{d^n}{dx^n} + \frac{\nu}{x^2} (\frac{1}{2} - \nu^2) f(x)$, $n = 0, 1, \ldots$, belong to $L_2(R_+)$;

iii) $\frac{d^n}{dx^n} + \frac{\nu}{x^2} (\frac{1}{2} - \nu^2) f(x)$, $n = 0, 1, \ldots$, tend to 0 as $x$ tends to 0 and to infinity;

iv) $\frac{d^n}{dx^n} (\frac{d^n}{dx^n} + \frac{\nu}{x^2} (\frac{1}{2} - \nu^2) f(x)$, $n = 0, 1, \ldots$, tend to 0 as $x$ tends to infinity and are bounded at 0;

Proof. Necessary. Let $y^n g(y)$ belong to $L_2(R_+)$ for all $n = 0, 1, 2, \ldots$, then $y^n g(y)$ belong to $L_1(R_+)$ for all $n = 0, 1, 2, \ldots$. Let $f(x)$ be the Hankel transform $\mathcal{H}_\nu$ of $g(y)$.

i) We have [1]

$$\frac{d^n}{dx^n} J_\nu(x) = 2^{-n} \sum_{j=0}^{n} (-1)^j \binom{n}{j} J_{\nu-n+2j}(x).$$  \hspace{1cm} (13)

Hence,

$$\frac{d^n}{dx^n} \sqrt{x} y J_\nu(xy) = \sum_{k=0}^{n} \sum_{j=0}^{k} (-1)^{n+j-k} 2^{-k} (-1/2)^{n-k} \binom{n}{k} \binom{k}{j} x^{1/2+k-n_1/2+k} J_{\nu-k+2j}(xy).$$  \hspace{1cm} (14)

Here $(a)_n$ is the Pochhammer symbol defined by $(a)_n = \Gamma(a+n)/\Gamma(a)$. The Bessel function of the first kind $J_\nu(y)$ has the asymptotics [1]

$$J_\nu(y) = \left\{ \begin{array}{ll} \sqrt{\frac{\pi y}{2}} \cos \left( y - \frac{\nu \pi}{2} - \frac{\pi}{4} \right) + \frac{1-4y^2}{8y} \sin \left( y - \frac{\nu \pi}{2} - \frac{\pi}{4} \right) + O(y^{-2}) , & y \to \infty, \\ O(y) , & y \to 0 \end{array} \right. \hspace{1cm} (15)$$
Consequently, \( \frac{\partial n}{\partial x} [\sqrt{xy} J_\nu(xy)] \) as a function of \( y \) has the order \( O(y^{1/2 + \Re \nu}) \) in the neighbourhood of 0 and \( O(y^n) \) at infinity. Hence, \( \frac{\partial n}{\partial x} [\sqrt{xy} J_\nu(xy)]g(y) \), \( \Re \nu > -1 \), as a function of \( y \) belong to \( L_1(R_+) \) for all \( n = 0, 1, 2, \ldots \). Therefore, \( f(x) \) is infinitely differentiable on \( R_+ \).

ii) Since \( J_\nu(x) \) satisfies the differential equation [1]
\[
x^2 u'' + xu' + (x^2 - \nu^2)u = 0,
\]
then \( \sqrt{x} J_\nu(x) \) is a solution of the equation
\[
x^2 u'' + \left( x^2 + \frac{1}{4} - \nu^2 \right) u = 0.
\]
Therefore, we have
\[
\left[ \frac{\partial^2}{\partial x^2} + \frac{1}{x^2} \left( \frac{1}{4} - \nu^2 \right) \right]^n (\sqrt{xy} J_\nu(xy)) = (-y^2)^n \sqrt{xy} J_\nu(xy).
\]
Consequently,
\[
\left[ \frac{d^2}{dx^2} + \frac{1}{x^2} \left( \frac{1}{4} - \nu^2 \right) \right]^n f(x) = (-1)^n \int_0^\infty \sqrt{xy} J_\nu(xy)g^2n g(y)dy, \quad \Re \nu > -1.
\]

Because of the Plancherel’s inequality (12) and \( y^2 n g(y) \in L_2(R_+) \) we obtain that \( \left[ \frac{d^2}{dx^2} + \frac{1}{x^2} \left( \frac{1}{4} - \nu^2 \right) \right]^n f(x), \quad \Re \nu > -1, \ n = 0, 1, \ldots \), belong to \( L_2(R_+) \).

iii) For the kernel \( \sqrt{xy} J_\nu(xy) \) has the asymptotics \( x^{\nu+1/2} \) as \( x \) tends to 0, is uniformly bounded on \( (0, \infty) \) if \( \Re \nu \geq -1/2 \), and \( y^2 n g(y) \in L_1(0, \infty) \), then applying the dominated convergence theorem we have
\[
\lim_{x \to 0} \left[ \frac{d^2}{dx^2} + \frac{1}{x^2} \left( \frac{1}{4} - \nu^2 \right) \right]^n f(x) = (-1)^n \int_0^\infty \sqrt{xy} J_\nu(xy)g^2n g(y)dy = 0, \quad \Re \nu > -1/2.
\]
Reasoning the same way, for every \( \epsilon > 0 \) one can choose \( N \) large enough so that
\[
\left| \int_N^\infty \sqrt{xy} J_\nu(xy)g^2n g(y)dy \right| < \epsilon,
\]
uniformly with respect to \( x \in R_+ \). The Bessel function \( J_\nu(y) \) has the asymptotics (15), therefore, the integral
\[
\int_{ax}^{bx} \sqrt{y} J_\nu(y)dy, \quad \Re \nu \geq -1/2,
\]
is uniformly bounded for all non-negative \( a, b \) and \( x \). Hence
\[
\int_a^b \sqrt{y} J_\nu(xy)dy = \frac{1}{x} \int_{ax}^{bx} \sqrt{y} J_\nu(y)dy, \quad \Re \nu \geq -1/2,
\]

4
tends to 0 uniformly in \(a, b\) for \(0 \leq a < b < \infty\) as \(x\) tends to infinity. Consequently, applying the generalized Riemann-Lebesgue theorem \([27]\) we get

\[
\lim_{x \to \infty} \int_0^N \sqrt{xy} J_\nu(xy) y^{2n} g(y) dy = 0, \quad 0 < N < \infty, \quad \Re \nu \geq -1/2.
\] (24)

Because \(\epsilon\) can be taken arbitrarily small, from (21) and (24) we have

\[
\lim_{x \to \infty} \int_0^\infty \sqrt{xy} J_\nu(xy) y^{2n} g(y) dy = 0, \quad \Re \nu \geq -1/2.
\] (25)

Hence,

\[
\lim_{x \to \infty} \left[ \frac{d^2}{dx^2} + \frac{1}{x^2} \left( \frac{1}{4} - \nu^2 \right) \right]^n f(x) = 0, \quad n = 0, 1, \ldots, \quad \Re \nu \geq -1/2. \tag{26}
\]

iv) Using the formula \([1]\)

\[
\frac{\partial}{\partial x} [\sqrt{xy} J_\nu(xy)] = \frac{1}{2} \sqrt{\pi} J_\nu(xy) + \frac{y}{2} \sqrt{\pi} y (J_{\nu-1}(xy) - J_{\nu+1}(xy))
\] (27)

we have

\[
(-1)^n \frac{d}{dx} \left[ \frac{d^2}{dx^2} + \frac{1}{x^2} \left( \frac{1}{4} - \nu^2 \right) \right]^n f(x) = \frac{1}{2x} \int_0^\infty \sqrt{xy} J_\nu(xy) y^{2n} g(y) dy \\
+ \frac{1}{2} \int_0^\infty \sqrt{xy} J_{\nu-1}(xy) y^{2n+1} g(y) dy - \frac{1}{2} \int_0^\infty \sqrt{xy} J_{\nu+1}(xy) y^{2n+1} g(y) dy. \tag{28}
\]

From iii) and especially from (20) and (25) we see that when \(\Re \nu \geq 1/2\) the first and the second expressions on the right hand side of (28) are uniformly bounded on \(R_+\), and, in particular, at 0. Hence, \(\frac{d}{dx} \left[ \frac{d^2}{dx^2} + \frac{1}{x^2} \left( \frac{1}{4} - \nu^2 \right) \right]^n f(x), \quad \Re \nu \geq 1/2\), tends to 0 at infinity and is bounded at 0.

**Sufficiency)** Suppose now that \(f\) satisfies the conditions i)-iv) of the theorem. Then \([\frac{d^2}{dx^2} + \frac{1}{x^2} \left( \frac{1}{4} - \nu^2 \right)]^n f(x)\) belongs to \(L_2(R_+)\) for all \(n = 0, 1, \ldots\). Let \(g_n(y)\) be its Hankel transform, that means

\[
g_n(y) = \int_0^\infty \sqrt{xy} J_\nu(xy) \left[ \frac{d^2}{dx^2} + \frac{1}{x^2} \left( \frac{1}{4} - \nu^2 \right) \right]^n f(x) dx, \quad \Re \nu \geq 1/2, \quad n = 0, 1, 2, \ldots, \tag{29}
\]

where the integral is understood in \(L_2\) sense. Putting

\[
g_n^N(y) = \int_{1/N}^N \sqrt{xy} J_\nu(xy) \left[ \frac{d^2}{dx^2} + \frac{1}{x^2} \left( \frac{1}{4} - \nu^2 \right) \right]^n f(x) dx, \quad n = 0, 1, 2, \ldots, \tag{30}
\]
we see that \( g_n^N(y) \) tends to \( g_n(y) \) in \( L_2 \) norm as \( N \rightarrow \infty \). Let \( n \geq 1 \). Integrating (30) by parts twice we obtain

\[
g_n^N(y) = \left\{ \sqrt{x} y J_\nu(xy) \frac{d}{dx} \left[ \frac{d^2}{dx^2} + \frac{1}{x^2} \left( \frac{1}{4} - \nu^2 \right) \right]^{n-1} f(x) \right\}_{x=1}^{x=N} - \left\{ \frac{\partial}{\partial x} \left( \sqrt{x} y J_\nu(xy) \right) \left[ \frac{d^2}{dx^2} + \frac{1}{x^2} \left( \frac{1}{4} - \nu^2 \right) \right]^{n-1} f(x) \right\}_{x=1/N}^{x=N} + \int_{1/N}^{N} \frac{\partial^2}{\partial x^2} + \frac{1}{x^2} \left( \frac{1}{4} - \nu^2 \right) \left( \sqrt{x} y J_\nu(xy) \right) \left[ \frac{d^2}{dx^2} + \frac{1}{x^2} \left( \frac{1}{4} - \nu^2 \right) \right]^{n-1} f(x) dx. \quad (31)
\]

Using formulas (18) and (27) we get

\[
g_n^N(y) = \sqrt{Ny} J_\nu(Ny) \frac{d}{dx} \left[ \frac{d^2}{dx^2} + \frac{1}{x^2} \left( \frac{1}{4} - \nu^2 \right) \right]^{n-1} f(N) \quad (32)
\]

\[
- \frac{N}{N} y J_\nu(y/N) \frac{d}{dx} \left[ \frac{d^2}{dx^2} + \frac{1}{x^2} \left( \frac{1}{4} - \nu^2 \right) \right]^{n-1} f(1/N) \quad (33)
\]

\[
- \frac{1}{2} \sqrt{N} y J_\nu(Ny) \left[ \frac{d^2}{dx^2} + \frac{1}{x^2} \left( \frac{1}{4} - \nu^2 \right) \right]^{n-1} f(N) \quad (34)
\]

\[
- \frac{1}{2} y \sqrt{Ny} J_\nu-1(Ny) \left[ \frac{d^2}{dx^2} + \frac{1}{x^2} \left( \frac{1}{4} - \nu^2 \right) \right]^{n-1} f(N) \quad (35)
\]

\[
+ \frac{1}{2} y \sqrt{Ny} J_\nu+1(Ny) \left[ \frac{d^2}{dx^2} + \frac{1}{x^2} \left( \frac{1}{4} - \nu^2 \right) \right]^{n-1} f(N) \quad (36)
\]

\[
+ \frac{1}{2} \sqrt{Ny} J_\nu(y/N) \left[ \frac{d^2}{dx^2} + \frac{1}{x^2} \left( \frac{1}{4} - \nu^2 \right) \right]^{n-1} f(1/N) \quad (37)
\]

\[
+ \frac{1}{2} y \sqrt{N} J_\nu-1(y/N) \left[ \frac{d^2}{dx^2} + \frac{1}{x^2} \left( \frac{1}{4} - \nu^2 \right) \right]^{n-1} f(1/N) \quad (38)
\]

\[
- \frac{1}{2} y \sqrt{N} J_\nu+1(y/N) \left[ \frac{d^2}{dx^2} + \frac{1}{x^2} \left( \frac{1}{4} - \nu^2 \right) \right]^{n-1} f(1/N) \quad (39)
\]

\[
y^2 \int_{1/N}^{N} \sqrt{x} y J_\nu(xy) \left[ \frac{d^2}{dx^2} + \frac{1}{x^2} \left( \frac{1}{4} - \nu^2 \right) \right]^{n-1} f(x) dx. \quad (40)
\]

Here \( P \left( \frac{d}{dx} \right) f(N) \) means \( P \left( \frac{d}{dx} \right) f(x) \mid_{x=N} \).

Since \( \sqrt{Ny} J_\nu(Ny) \) is uniformly bounded and \( \frac{d}{dx} \left[ \frac{d^2}{dx^2} + \frac{1}{x^2} \left( \frac{1}{4} - \nu^2 \right) \right]^{n-1} f(N) \) tends to 0 as \( N \) approaches infinity (property iv)), the expression on the right hand side of (32) tends to 0 as \( N \) approaches infinity.

Using property iv) we see that \( \frac{d}{dx} \left[ \frac{d^2}{dx^2} + \frac{1}{x^2} \left( \frac{1}{4} - \nu^2 \right) \right]^{n-1} f(1/N) \) is bounded, whereas function \( \sqrt{N} J_\nu(y/N) \) has the order \( O(N^{-\nu-1/2}) \) at infinity. Hence, expression (33) tends
to 0 as \( N \) approaches infinity. Similarly, function \( \frac{1}{2} \sqrt{\frac{N}{2}} J_{\nu}(Ny) \) has the order \( O(N^{-1}) \) and 
\[
\left[ \frac{d^2}{dx^2} + \frac{1}{x^2} \left( 1 - \nu^2 \right) \right]^{n-1} f(N) \text{ is } o(1) \text{ (property iii))}, \text{ therefore, expression (34) is } o(1).
\]

Functions \( \frac{1}{2} y \sqrt{\frac{N}{2}} J_{\nu-1}(Ny) \) and \( \frac{1}{2} y \sqrt{\frac{N}{2}} J_{\nu+1}(Ny) \) both are \( o(1) \), and function 
\[
\left[ \frac{d^2}{dx^2} + \frac{1}{x^2} \left( 1 - \nu^2 \right) \right]^{n-1} f(N) \text{ is also } o(1) \text{ (property iii))}, \text{ therefore, both expressions (35) and (36) tend to 0 as } N \text{ tends to infinity.}
\]

Functions \( \frac{1}{2} \sqrt{\frac{N}{2}} J_{\nu}(Ny/N) \), \( \frac{1}{2} y \sqrt{\frac{N}{2}} J_{\nu-1}(y/N) \), and \( \frac{1}{2} y \sqrt{\frac{N}{2}} J_{\nu+1}(y/N) \) are bounded, whereas function 
\[
\left[ \frac{d^2}{dx^2} + \frac{1}{x^2} \left( 1 - \nu^2 \right) \right]^{n-1} f(1/N) \text{ is } o(1) \text{ as } N \text{ tends to infinity (property iii))}, \text{ hence, all the expressions (37), (38) and (39) tend to 0 as } N \text{ tends to infinity.}
\]

Function (40) converges to \(-y^2 g_{n-1}(y)\) as \( N \) approaches infinity, hence, \( g_n(y) = -y^2 g_{n-1}(y) \), and therefore, \( g_n(y) = (-y^2)^n g_0(y) \), \( n = 0, 1, \ldots \). But if \( g \) is the Hankel transform (2) of \( f \), then \( f \) is the Hankel transform (1) of \( g \). Therefore, \( f(x) \) is the Hankel transform of a function \( g(y) = g_0(y) \) such that \( y^{2n} g(y) \in L_2(R_+) \), \( n = 0, 1, \ldots \), and Theorem 1 is thus proved.

**Corollary 1.** The Zemanian space \( H_\nu \), \( \Re \nu \geq 1/2 \), can be characterized as the set of functions \( f(x) \), satisfying conditions i)-iv) of Theorem 1 and such that \( x^n f(x) \in L_2(R_+) \), \( n = 0, 1, 2, \ldots \).

**Proof.** It is well known [5] that \( f \in H_\nu \) if and only if \( f(x) = x^{\nu+1/2} \phi(x) \), where \( \phi \in S_{\text{even}} \), the set of restrictions of even Schwartz functions on \( R_+ \). It is proved [8] that \( \phi \in S_{\text{even}} \) if and only if
\[
\sup_{x \in R_+} |x^n \phi(x)| < \infty, \sup_{x \in R_+} |x^n \phi(x)| < \infty, n = 0, 1, \ldots \quad (41)
\]
where
\[
\phi(x) = \int_0^\infty (xy)^{-\nu} J_\nu(xy) \phi(y) y^{2\nu+1} dy. \quad (42)
\]
Hence \( f \in H_\nu \) if and only if
\[
\sup_{x \in R_+} |x^{n-\nu-1/2} f(x)| < \infty, \sup_{x \in R_+} |x^n \mathcal{H}_\nu f(x)| < \infty, n = 0, 1, \ldots \quad (43)
\]
Suppose that \( f \) satisfies conditions i)-iv) of Theorem 1 and \( x^n f(x) \in L_2(R_+) \), \( n = 1, 2, \ldots \). Then \( x^{-\nu-1/2} f(x) \) is bounded at 0. Since \( x^n f(x) \) tends to 0 at 0 and infinity, \( x^{n-\nu-1/2} f(x) \) is bounded on \( R_+ \). Since the Hankel transform has the symmetric inverse, \( x^{n-\nu-1/2} \mathcal{H}_\nu f(x) \) is also bounded on \( R_+ \). But it is equivalent to the fact that \( f \in H_\nu \).

Suppose now that \( f \in H_\nu \). Then \( \mathcal{H}_\nu f \in H_\nu \). Hence, inequality (10) is valid with \( k = 0 \) for both \( f \) and \( \mathcal{H}_\nu f \):
\[
|x^{n-\nu-1/2} f(x)| < \infty, |x^{n-\nu-1/2} \mathcal{H}_\nu f(x)| < \infty, n = 0, 1, \ldots \quad (44)
\]
Hence, \( f \) satisfies conditions i)-iv) of Theorem 1 and \( x^n f(x) \in L_2(R_+) \), \( n = 1, 2, \ldots \).

**3. HANKEL TRANSFORM OF SQUARE INTEGRABLE FUNCTIONS WITH COMPACT SUPPORTS**
Now we describe the Hankel transforms of square integrable functions with compact supports.

**Theorem 2.** (the Paley-Wiener theorem for the Hankel transform of square integrable functions with compact supports) A function \( f \) is the Hankel transform \( H_\nu \), \( \Re \nu \geq 1/2 \), of a square integrable function \( g \) with compact support on \([0, \infty)\) if and only if \( f \) satisfies conditions i)-iv) of Theorem 1 and moreover,

\[
\lim_{n \to \infty} \left\| \left[ \frac{d^2}{dx^2} + \frac{1}{x^2} \left( \frac{1}{4} - \nu^2 \right) \right]^n f(x) \right\|_2^{1/2n} = \sigma_g < \infty,
\]

where

\[
\sigma_g = \sup \{ y : y \in \text{supp } g \},
\]

and the support of a function is the smallest closed set, outside it the function vanishes almost everywhere [30].

**Proof.** a) Let \( f(x) \) be the Hankel transform of \( g(y) \in L_2(R_+) \) and \( \sigma_g < \infty \):

\[
f(x) = \int_0^{\sigma_g} \sqrt{xy} J_\nu(xy)g(y) \, dy, \quad \Re \nu \geq 1/2.
\]

One can assume that \( \sigma_g > 0 \), otherwise it is trivial. Since \( \sigma_g < \infty \), we have \( y^ng(y) \in L_2(R_+) \) for all \( n = 0, 1, 2, \ldots \). Therefore, \( f \) satisfies conditions i)-iv) of Theorem 1. Furthermore,

\[
\left[ \frac{d^2}{dx^2} + \frac{1}{x^2} \left( \frac{1}{4} - \nu^2 \right) \right]^n f(x) = \int_0^{\sigma_g} \sqrt{xy} J_\nu(xy)(-y^n)g(y) \, dy.
\]

Consequently, applying the right hand side inequality in (12) for the Hankel transform (48) we have

\[
\left\| \left[ \frac{d^2}{dx^2} + \frac{1}{x^2} \left( \frac{1}{4} - \nu^2 \right) \right]^n f(x) \right\|_2^2 \leq C \int_0^{\sigma_g} y^n |g(y)|^2 \, dy \leq C \sigma_g^{4n} \int_0^{\sigma_g} |g(y)|^2 \, dy.
\]

Hence,

\[
\lim_{n \to \infty} \left\| \left[ \frac{d^2}{dx^2} + \frac{1}{x^2} \left( \frac{1}{4} - \nu^2 \right) \right]^n f(x) \right\|_2^{1/(2n)} \leq \lim_{n \to \infty} C^{1/(4n)} \sigma_g \left\{ \int_0^{\sigma_g} |g(y)|^2 \, dy \right\}^{1/2n} = \sigma_g.
\]

On the other hand, since \( \sigma_g \) is the least upper bound of the support of \( g \), for every \( \epsilon, \ 0 < \epsilon < \sigma_g \), we have

\[
\int_{\sigma_g - \epsilon}^{\sigma_g} |g(y)|^2 \, dy > 0.
\]

Consequently, using the left hand side inequality in (12) we have

\[
\lim_{n \to \infty} \left\| \left[ \frac{d^2}{dx^2} + \frac{1}{x^2} \left( \frac{1}{4} - \nu^2 \right) \right]^n f(x) \right\|_2^{1/(2n)} \geq \lim_{n \to \infty} C^{-1/(4n)} \left\{ \int_{\sigma_g - \epsilon}^{\sigma_g} y^n |g(y)|^2 \, dy \right\}^{1/2n} \geq (\sigma_g - \epsilon) \lim_{n \to \infty} C^{-1/(4n)} \left\{ \int_{\sigma_g - \epsilon}^{\sigma_g} \left| g(y) \right|^2 \, dy \right\}^{1/2n} = \sigma_g - \epsilon.
\]
Because \( \epsilon \) can be chosen arbitrarily small, from (52) and (50) we obtain (45).

b) Suppose now that \( f \) satisfies the conditions i)-iv) of Theorem 1, and the limit in (45) exists and equals \( \sigma < \infty \). Using Theorem 1 we see that \( f \) is the Hankel transform of a function \( g \) such that \( g^n(y) \in L_2(R_+) \), \( n = 0, 1, 2, \ldots \). We shall prove that \( \sigma_g < \infty \), and moreover, \( \sigma = \sigma_g \). From Theorem 1 we have that (19) is valid. Therefore, applying the inequalities (12) we obtain

\[
C^{-1} \|y^{2n}g(y)\|_2 \leq \left\| \left[ \frac{d^2}{dx^2} + \frac{1}{x^2} \left( \frac{1}{4} - \nu^2 \right) \right]^n f(x) \right\|_2 \leq C \|y^{2n}g(y)\|_2. \tag{53}
\]

Hence,

\[
\lim_{n \to \infty} C^{-1/(2n)} \|y^{2n}g(y)\|_2^{1/(2n)} \leq \lim_{n \to \infty} \left\| \left[ \frac{d^2}{dx^2} + \frac{1}{x^2} \left( \frac{1}{4} - \nu^2 \right) \right]^n f(x) \right\|^{1/(2n)}_2 = \sigma \leq \lim_{n \to \infty} C^{1/(2n)} \|y^{2n}g(y)\|_2^{1/(2n)}.
\tag{54}
\]

Consequently,

\[
\lim_{n \to \infty} \|y^{2n}g(y)\|_2^{1/(2n)} = \sigma.
\tag{55}
\]

Suppose that \( \sigma_g > \sigma \). Then there exists a positive \( \epsilon \) such that

\[
\int_{\sigma + \epsilon}^{\infty} |g(y)|^2 dy > 0.
\tag{56}
\]

We have

\[
\sigma = \lim_{n \to \infty} \|y^{2n}g(y)\|_2^{1/(2n)} \geq \lim_{n \to \infty} \left\{ \int_{\sigma + \epsilon}^{\infty} y^{4n}|g(y)|^2 dy \right\}^{1/(4n)} \geq (\sigma + \epsilon) \lim_{n \to \infty} \left\{ \int_{\sigma + \epsilon}^{\infty} |g(y)|^2 dy \right\}^{1/(4n)} = \sigma + \epsilon, \tag{57}
\]

that is impossible. Hence \( \sigma_g \leq \sigma \) and therefore, \( g \) has a compact support.

Suppose now that \( \sigma_g < \sigma \). Then there exists a positive \( \epsilon \) such that

\[
\int_{\sigma - \epsilon}^{\infty} |g(y)|^2 dy = 0.
\tag{58}
\]

We have

\[
\sigma = \lim_{n \to \infty} \|y^{2n}g(y)\|_2^{1/(2n)} \leq \lim_{n \to \infty} \left\{ \int_{0}^{\sigma - \epsilon} y^{4n}|g(y)|^2 dy \right\}^{1/(4n)} \leq (\sigma - \epsilon) \lim_{n \to \infty} \left\{ \int_{0}^{\sigma - \epsilon} |g(y)|^2 dy \right\}^{1/(4n)} = \sigma - \epsilon, \tag{59}
\]

that is also impossible. Hence \( \sigma_g \geq \sigma \), and consequently, \( \sigma_g = \sigma < \infty \). Theorem 2 is thus proved.

**Remark 1.** From Theorems 1 and 2 it is not difficult to see that if a function \( f \) satisfies conditions i)-iv) of Theorem 1, then the limit (45) always exists. It equals infinity, if the Hankel transform of \( f \) has an unbounded support.
4. HANKEL TRANSFORM OF INFINITELY DIFFERENTIABLE FUNCTIONS WITH COMPACT SUPPORTS

Let the Erdelyi-Kober fractional integral operator $\mathcal{K}^\nu$ be defined by [25]

$$h_1(x) = (\mathcal{K}^\nu g_1)(x) = \int_x^\infty \frac{(y^2 - x^2)^{\nu - 1}}{\Gamma(\nu)} yg_1(y)\,dy, \quad \Re \nu > 0, \quad x \in \mathbb{R}. \quad (60)$$

We need the following

**Lemma 1.** The restriction of the Erdelyi-Kober fractional integral operator $\mathcal{K}^\nu$ (60) on $\mathbb{R}^+$ is a bijection on $S_{even}$ and a bijection on its subspace of functions with compact supports. Moreover, $\sigma_{h_1} = \sigma_{g_1}$.

**Proof.** It is proved [25] that the Weyl fractional integral operator

$$v(x) = (\mathcal{I}^-\nu u)(x) = \int_x^\infty \frac{(y - x)^{\nu - 1}}{\Gamma(\nu)} u(y)\,dy, \quad x \in \mathbb{R}, \quad (61)$$

is a bijection in the space of infinitely differentiable functions which have for $x \to \infty$ the same behaviour as functions in $S(\mathbb{R})$ and have a slow growth for $x \to -\infty$. It is known [14] that the Weyl fractional integral operator (61) is also a bijection on the space of infinitely differentiable functions with compact supports on $\mathbb{R}^+$. From (61) it is easy to see that $\sigma_v \leq \sigma_u$. Since the inverse of the Weyl fractional operator has the form [25]

$$u(x) = (-1)^m \frac{d^m}{dx^m} (\mathcal{I}^{-\nu}\nu)(x) = (-1)^m \frac{d^m}{dx^m} \int_x^\infty \frac{(y - x)^{m-\nu - 1}}{\Gamma(m - \nu)} v(y)\,dy, \quad m > \Re \nu, \quad (62)$$

one can get the reverse inequality $\sigma_u \leq \sigma_v$. Hence, $\sigma_u = \sigma_v$. On $\mathbb{R}^+$ formula (61) can be rewritten in the form (60) with $2u(y^2) = g_1(y)$, $(x^2) = h_1(x)$. One can prove that $u, v \in S(\mathbb{R})$ if and only if $g_1, h_1 \in S_{even}$ and $u, v$ have compact supports if and only if $g_1, h_1$ have compact supports, and moreover, $\sigma_u^2 = \sigma_{g_1}, \sigma_v^2 = \sigma_{h_1}$. Hence, if $\sigma_u = \sigma_v$, then $\sigma_{g_1} = \sigma_{h_1}$ and vice versa. Lemma 1 is just proved.

**Theorem 4.** (the Paley-Wiener-Schwartz theorem for the Hankel transform of infinitely differentiable functions with compact supports) A function $f \in H_\nu$ is the Hankel transform $\mathcal{H}_\nu$, $\Re \nu > -1/2$, of a function $g \in H_\nu$ with compact support if and only if

$$\sigma_g = \lim_{n \to \infty} \left\| \frac{d^m}{dx^m} x^{-\nu - 1/2} f(x) \right\|_p^{1/n} , \quad 1 \leq p \leq \infty. \quad (63)$$

**Proof.** The Bessel function $J_\nu(x)$ has the integral representation [1]

$$J_\nu(x) = \frac{2^{1-\nu} x^\nu}{\sqrt{\pi} \Gamma(\nu + 1/2)} \int_0^1 (1 - t^2)^{\nu - 1/2} \cos xt \, dt, \quad \Re \nu > -1/2. \quad (64)$$

Substituting $x$ by $xy$ and $t$ by $t/y$ we have

$$J_\nu(xy) = \frac{2^{1-\nu} x^\nu y^{-\nu}}{\sqrt{\pi} \Gamma(\nu + 1/2)} \int_0^y (y^2 - t^2)^{\nu - 1/2} \cos xt \, dt. \quad (65)$$
Consequently, the Hankel transform (1) can be rewritten in the form

\[ f(x) = \frac{2^{1-\nu}x^{\nu+1/2}}{\sqrt{\pi}\Gamma(\nu + 1/2)} \int_0^\infty y^{1/2-\nu}g(y) \int_y^\infty (y^2 - t^2)^{-\nu-1/2} \cos xt \, dt \, dy. \] (66)

If \( y^{\nu+1/2}g(y) \in L_1(R_+) \) then the repeated integral (66) converges absolutely. Therefore, one can apply the Fubini-Tonelli theorem [30] to interchange the order of integration in (66) to get

\[ f(x) = \frac{2^{1-\nu}x^{\nu+1/2}}{\sqrt{\pi}\Gamma(\nu + 1/2)} \int_0^\infty \cos xt \int_0^\infty (y^2 - t^2)^{-\nu-1/2} y^{\nu+1/2} g(y) \, dy \, dt. \] (67)

Putting \( y^{-\nu-1/2}g(y) = g_1(y) \) and \( x^{-\nu-1/2}f(x) = f_1(x) \), we have

\[ f_1(x) = \frac{2^{1-\nu}}{\sqrt{\pi}\Gamma(\nu + 1/2)} \int_0^\infty \cos xt \int_0^\infty (y^2 - t^2)^{-\nu-1/2} yg_1(y) \, dy \, dt. \] (68)

Therefore, \( f_1 \) can be viewed as a composition of the Fourier cosine transform

\[ f_1(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \cos xt \, h_1(t) \, dt, \quad 0 \leq x < \infty, \] (69)

and the Erdelyi-Kober fractional integral operator \( K^{\nu+1/2} \) (60) of order \( \nu + 1/2 \), multiplied by a constant. From [2] we see that \( \hat{f} \in S(R) \) is the Fourier transform of an infinitely differentiable function \( f \) on \( R \) with compact support if and only if

\[ \sigma_{|f|} = \lim_{n \to \infty} \left\| \frac{d^n}{dx^n} \hat{f}(x) \right\|_{L_p(R)}^{1/n}, \quad 1 \leq p \leq \infty, \] (70)

where

\[ \sigma_{|f|} = \sup \{|y| : y \in \text{supp } f\}. \] (71)

Restricting the Fourier transform only on even functions we see that a function \( f_1 \in S_{\text{even}} \) is the Fourier cosine transform (69) of a function \( h_1 \in S_{\text{even}} \) with compact support if and only if

\[ \sigma_{h_1} = \lim_{n \to \infty} \left\| \frac{d^n}{dx^n} f_1(x) \right\|_{p}^{1/n}. \] (72)

On the other hand, the Erdelyi-Kober fractional integral operator \( K^{\nu+1/2} \) (60) is a bijection in the space of infinitely differentiable functions on \( \mathbb{R}_+ \) with compact supports and \( \sigma_{h_1} = \sigma_{g_1} \) (Lemma 1). Remarking that if \( g_1(y) = y^{-\nu-1/2}g(y) \) then \( \sigma_{g_1} = \sigma_{g} \), Theorem 3 follows now from formula (68).

Remark 3. Theorem 2 and 3 involve the spectral radii [30] of some differential operators (see formulas (45) and (63)), but the proofs are obtained straightforward, without referring to the spectral theory. In [14] the Hankel transform of infinitely differentiable functions with compact supports has been described by classical way through entire functions of exponential type.
5. HANKEL AND EXTENDED HANKEL TRANSFORM OF ANALYTIC
FUNCTIONS

Let $\mathcal{H}_G$ be the space of functions $g(z)$ that are
- regular in the angle $-\alpha < \arg z < \beta$, where $0 < \alpha, \beta \leq \pi$;
- of the order $O(|z|^{-a+\epsilon})$ for small $z$, and $O(|z|^{-b+\epsilon})$ for large $z$, where $a < 1/2 < b$,
uniformly in any angle interior to the above for any small positive $\epsilon$;
- satisfying the condition
$$
\int_0^\infty y^{\nu+2n+1/2} g(y) \, dy = 0, \quad (73)
$$
for all nonnegative integers $n$ on the interval $(a/2 - \Re \nu/2 - 1/4, b/2 - \Re \nu/2 - 1/4)$, provided there exists one.

Let $\mathcal{H}_F$ be Theorem 4. The Hankel transform (1) and the extended Hankel transform (3) map, one-to-one onto the space of functions $f(z)$, regular in the angle $-\beta < \arg z < \alpha$, of the order $O(|z|^{-1-b+\epsilon})$ for small $z$, and $O(|z|^{-1+a-\epsilon})$ for large $z$, uniformly in any angle interior to the above for any small positive $\epsilon$, and satisfying conditions
$$
\int_0^\infty x^{\nu+2n+1/2} f(x) \, dx = 0, \quad (74)
$$
for all nonnegative integers $n$ on the interval $(-b/2 - \Re \nu/2 - 1/4, -a/2 - \Re \nu/2 - 1/4)$.

Proof. Let $g(z)$ satisfy the conditions of Theorem 5. Then function $g(z)$ on $R_+$ belongs
to $L_2(R_+)$ and its Mellin transform $g^*(s)$, defined by [27]
$$
g^*(s) = \int_0^\infty x^{s-1} g(x) \, dx, \quad \Re s = 1/2, \quad (75)
$$
is an analytic function of $s$, regular for $a < \Re s < b$; and
$$
g^*(s) = \begin{cases} 
O\left(e^{-(\beta-\epsilon)\Im s}\right), & \Im s \to \infty \\
O\left(e^{(\alpha-\epsilon)\Im s}\right), & \Im s \to -\infty 
\end{cases}, \quad (76)
$$
for every positive $\epsilon$, uniformly in any strip interior to $a < \Re s < b$ (see [27]). Let,
furthermore, $f(x)$ be the Hankel transform (1) or the extended Hankel transform (3) of
$g(y)$ and $f^*(s)$ be its Mellin transform (75). Since $g(y)$ belongs to $L_2(R_+)$ the Hankel and
extended Hankel transforms in $L_2(R_+)$ are equivalent to the following Parseval equation
for $f^*(s)$ and $g^*(s)$ in $L_2(1/2 - i\infty, 1/2 + i\infty)$ on the line $\Re s = 1/2$ [27, 13]:
$$
f^*(s) = 2^{s-1/2} \frac{\Gamma(\nu/2 + s/2 + 1/4)}{\Gamma(\nu/2 - s/2 + 3/4)} g^*(1-s). \quad (77)
$$
Because of (73) the function $g^*(1-s)$ equals $0$ at the poles of the gamma function $\Gamma(\nu/2 + s/2 + 1/4)$ in the strip $1 - b < \Re s < 1 - a$, if there exists one. Hence, from (77) one
can see that $f^*(s)$ is analytic in the strip $1 - b < \Re s < 1 - a$. Furthermore, since
the function $2^{s-1/2} \frac{\Gamma(\nu/2 + s/2 + 1/4)}{\Gamma(\nu/2 - s/2 + 3/4)}$ is uniformly bounded on any compact domain in the strip
1 − b < \Re s < 1 − a, containing no poles of the function \Gamma(\nu/2 + s/2 + 1/4), and has at most polynomial growth at infinity, from (76) we see that the function \phi^*(s) also decays exponentially
\[
\phi^*(s) = \begin{cases} 
O(e^{(\beta-\epsilon)\Im s}), & \Im s \to -\infty \\
O(e^{-(\alpha-\epsilon)\Im s}), & \Im s \to \infty,
\end{cases}
\tag{78}
\]
for every positive \epsilon, uniformly in any strip interior to 1 − b < \Re s < 1 − a. Hence, its inverse Mellin transform \phi(z) is regular for −\beta < \arg z < \alpha, and of the order \O(|z|^{1-\epsilon}) for small z, and \O(|z|^{\alpha+\epsilon}) for large z, uniformly in any angle interior to the above angle [27]. Moreover, \phi^*(s) has zeros at poles of the gamma function \Gamma(\nu/2 - s/2 + 3/4) in the strip 1 − b < \Re s < 1 − a, if there exists one, therefore, (74) is satisfied.

Conversely, let \phi(z) satisfy the conditions of Theorem 4. Then \phi(x) belongs to \L_2(R_+) and its Mellin transform \phi^*(s) is analytic in the strip 1 − b < \Re s < 1 − a, has zeros at poles of the gamma function \Gamma(\nu/2 - s/2 + 3/4) in the strip 1 − b < \Re s < 1 − a, if there exists one, and satisfies (78). Therefore, if we express \phi^*(s) in the form (77), function \psi^*(s) is analytic in the strip \alpha < \Re s < \beta; and has the exponential decay (76) for every positive \epsilon, uniformly in any strip interior to \alpha < \Re s < \beta. Furthermore, \psi^*(1 - s) has zeros at poles of the gamma function \Gamma(\nu/2 + s/2 + 1/4) in the strip \alpha < \Re s < \beta, hence, formula (73) is valid for all nonnegative integers \nu on the interval (\alpha/2 - \Re \nu/2 - 1/4, \beta/2 - \Re \nu/2 - 1/4), if there exists such \nu.

If in Theorem 5 we take \beta = \alpha and 1/2 − a = b − 1/2, then the spaces of functions \phi and \psi coincide, hence, denoting 1/2 − a again by a we have

**Corollary 2.** The Hankel transform (1) and the extended Hankel transform (3) are bijections in the space of functions \phi(z), regular in the angle |\arg z| < \alpha, where 0 < \alpha \leq \pi; of the order \O(|z|^{-1/2+a-\epsilon}) for small z, and \O(|z|^{-1/2-a+\epsilon}) for large z, where \alpha > 0, uniformly in any angle interior to the above for any small positive \epsilon, 0 < \epsilon < a, and satisfying conditions (73) for all nonnegative integers \nu on the interval (−\alpha/2 − \Re \nu/2, \alpha/2 − \Re \nu/2), if there exists such \nu.

6. HANKEL AND EXTENDED HANKEL TRANSFORM IN SOME OTHER SPACES OF FUNCTIONS

Let \Phi be any linear symmetric subspace of \L_1(R) or \L_2(R) having \mu(t) = 2^\nu \Gamma(\nu/2 + it/2 + 1/2)/\Gamma(\nu/2 - it/2 + 1/2), \Re \nu \neq -1, -3, \ldots, as multiplier (symmetric means that if \phi(t) \in \Phi then \phi(-t) \in \Phi). The multiplier \mu(t) \Gamma(\nu/2 + 1/2 + it/2)/\Gamma(\nu/2 + 1/2 - it/2) is infinitely differentiable and uniformly bounded on \R, its derivatives grow slowly, therefore many classical spaces on \R are special cases of \Phi (for example, any \L_p space with \L^p-weights, space \S(R) and space of infinitely differentiable functions with compact supports [30]). We define by \M^{-1}(\Phi) the space of all functions \phi on \R_+ that can be represented in the form
\[
\phi(x) = \int_{-\infty}^{\infty} \phi(t)x^{it-1/2}dt,
\tag{79}
\]

almost everywhere, where \( \phi \in \Phi \) (if \( \phi \notin L_1(R) \) the integral should be understood in \( L_2 \)-meaning). The space \( M^{-1}_\phi(L) \) [28] as well as the space of functions considered in Corollary 2 are special cases of \( M^{-1}(\Phi) \).

**Theorem 5.** The Hankel transform (1) and the extended Hankel transform (3) are bijections in \( M^{-1}(\Phi) \).

**Proof.** From (79) we see that \( g \in M^{-1}(\Phi) \) if and only if \( g \) can be expressed in the form of the inverse Mellin transform [27]

\[
g(x) = \frac{1}{2\pi i} \int_{1/2+i\infty}^{1/2-i\infty} g^*(s)x^{-s}ds,
\]

where \( g^*(1/2 + it) \in \Phi \). Let \( g^*(s) \) belong either to \( L_1(1/2 - i\infty, 1/2 + i\infty) \) or \( L_2(1/2 - i\infty, 1/2 + i\infty) \), and \( g(x) \) be its inverse Mellin transform. It is proved (see [27] for the case \( g^*(s) \in L_1(1/2 - i\infty, 1/2 + i\infty) \) and [28] for the case \( g^*(s) \in L_1(1/2 - i\infty, 1/2 + i\infty) \)) that under these assumptions the Parseval formula for the Hankel transform

\[
(\mathcal{H}_{\nu}g)(x) = \int_0^\infty \sqrt{xy}J_{\nu}(xy)g(y)dy
\]

holds. The Parseval formula (81) is also valid, if we replace the Bessel function \( J_\nu(x) \) by the truncated Bessel function \( J_{\nu,m}(x) \) in case \( 1 - 2m > \Re \nu > -2m - 1 \):

\[
(\mathcal{H}_{\nu}g)(x) = \int_0^\infty \sqrt{xy}J_{\nu,m}(xy)g(y)dy
\]

**Proof.** From (79) we see that \( g \in M^{-1}(\Phi) \) if and only if \( g \) can be expressed in the form of the inverse Mellin transform [27]

\[
g(x) = \frac{1}{2\pi i} \int_{1/2+i\infty}^{1/2-i\infty} g^*(s)x^{-s}ds,
\]

where \( g^*(1/2 + it) \in \Phi \). Let \( g^*(s) \) belong either to \( L_1(1/2 - i\infty, 1/2 + i\infty) \) or \( L_2(1/2 - i\infty, 1/2 + i\infty) \), and \( g(x) \) be its inverse Mellin transform. It is proved (see [27] for the case \( g^*(s) \in L_1(1/2 - i\infty, 1/2 + i\infty) \) and [28] for the case \( g^*(s) \in L_1(1/2 - i\infty, 1/2 + i\infty) \)) that under these assumptions the Parseval formula for the Hankel transform

\[
(\mathcal{H}_{\nu}g)(x) = \int_0^\infty \sqrt{xy}J_{\nu}(xy)g(y)dy
\]

holds. The Parseval formula (81) is also valid, if we replace the Bessel function \( J_\nu(x) \) by the truncated Bessel function \( J_{\nu,m}(x) \) in case \( 1 - 2m > \Re \nu > -2m - 1 \):

\[
(\mathcal{H}_{\nu}g)(x) = \int_0^\infty \sqrt{xy}J_{\nu,m}(xy)g(y)dy
\]

In fact, it is true if \( g(x) \in L_2(R) \), or equivalently, \( g^*(s) \in L_2(1/2 - i\infty, 1/2 + i\infty) \).

Let now \( g^*(s) \in L_1(1/2 - i\infty, 1/2 + i\infty) \). From (4) we see that \( J_{\nu,m}(x) = O(x^{\nu+2m}) \) at 0, therefore, integral \( \int_0^1 x^{s-1/2}J_{\nu,m}(x)dx \) converges absolutely if \( \Re s = 1/2 \). Integral \( \int_1^\infty x^{s-1/2} \sum_{k=0}^{m-1} \frac{(-1)^k(\frac{s}{2})^{\nu+2k}}{\Gamma(\nu+k+1)} dx \) also converges absolutely. Applying the asymptotics (15) of the Bessel function one can easily see that integral \( \int_1^N x^{s-1/2}J_\nu(x)dx \) is uniformly bounded for all \( N \in [1, \infty) \) and \( s \in (1/2 - i\infty, 1/2 + i\infty) \). Hence, integral

\[
\int_0^N x^{s-1/2}J_{\nu,m}(x)dx
\]

is uniformly bounded for all \( N \in [1, \infty) \) and \( s \in (1/2 - i\infty, 1/2 + i\infty) \). Consequently, applying the result from [28] we get (82).

Because \( m^{-1}(-t) = m(t) \), then \( 2\Gamma(\nu/2+u/2+1/2)g^*(1/2 - it) \) belongs to \( \Phi \) if and only if \( g^*(1/2 + it) \) belongs to \( \Phi \). Hence, from (81) and (82) we obtain that \( (\mathcal{H}_{\nu}g)(x) \) (the Hankel or extended Hankel transform of \( g \)) belongs to \( M^{-1}(\Phi) \) if and only if \( g(x) \in M^{-1}(\Phi) \). Theorem 5 is proved.
References


DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, KUWAIT UNIVERSITY, P.O. BOX 5969, SAFAT 13060, KUWAIT
E-mail address: vu@math-1.sci.kuwait.edu.kw