ON THE GENERALIZED CONVOLUTIONS FOR FOURIER COSINE AND SINE TRANSFORMS

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Abstract. A generalized convolution for the Fourier cosine and sine transforms is introduced, its properties and applications to integral equations are considered.

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1. Introdution

Generalized convolution of functions f and g under three operators K, K_1, K_2, and with some weight-function γ is a function, denoted by the symbol $f * g$, such that the following factorization property holds [5] :

$$K(f * g)(x) = \gamma(x)(K_1 f)(x)\,(K_2 g)(x). \tag{1.1}$$

If $K = K_1 = K_2$ we have the usual classical convolution [3,4]. For example, for $K = K_1 = K_2 = F_c$ – the Fourier cosine transform [8]

$$(F_c f)(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(y) \cos xy \, dy, \tag{1.2}$$

the convolution has the form [8]

$$(f \overset{0}{*} g)(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(y)[g(\mid x - y \mid) + g(x+y)]dy, \tag{1.3}$$

and the property (1.1) holds

$$F_c(f \overset{0}{*} g)(x) = (F_c f)(x)\,(F_c g)(x). \tag{1.4}$$

Otherwise, there appear "exotic" convolutions. An example of generalized convolutions was first introduced by Churchill [1]

$$(f \overset{1}{*} g)(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(y)[g(\mid x - y \mid) - g(x+y)]dy, \tag{1.5}$$

and the respective factorization property (1.1) for (1.5) has the form

$$F_s(f \overset{1}{*} g)(x) = (F_s f)(x)\,(F_c g)(x), \tag{1.6}$$

where F_s is the Fourier sine transform [8]

$$(F_s f)(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(y) \sin\, xy\, dy. \tag{1.7}$$

Many authors have been studied similar convolutions for Hankel's transform [10], Stieltjes' transform [9], Hilbert's transform [2], G-transform [7], and integral transforms of Mellin convolution type [6,12]. The present work is devoted to investigate properties of another generalized convolution for Fourier cosine and sine transforms, different from (1.5), and its application to a linear system of integral equations.

2. The generalized convolution

Definition 1. *A generalized convolution for the Fourier cosine and sine transforms is defined as follows*:

$$(f \overset{2}{*} g)(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(y)[\mathrm{sign}(y-x) g(\mid y - x \mid) + g(y+x)]dy. \tag{2.1}$$

Theorem 1. *Let* $f, g \in L(R_+)$, *then the convolution* $f \overset{2}{*} g$ *belongs to* $L(R_+)$ *and*

$$F_c(f \overset{2}{*} g)(x) = (F_s f)(x)\,(F_s g)(x), \qquad x \epsilon R_+ \ . \tag{2.2}$$

Proof. We have

$$\int_0^\infty |\,(f \overset{2}{*} g)(x)\,|\,dx \leq \frac{1}{\sqrt{2\pi}} \int_0^\infty \int_0^\infty |\,(f(y)\,|\,(|\,g(|\,x-y\,|)\,|+|\,g(x+y)\,|)dxdy$$

$$\leq \frac{1}{\sqrt{2\pi}} \int_0^\infty |\,f(y)\,|\,dy\ \left[\int_{-y}^\infty |\,g(|x|)\,|\,dx + \int_y^\infty |\,g(x)\,|\,dx\right]$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty |\,f(y)\,|\,dy\ \int_0^\infty |\,g(x)\,|\,dx < \infty.$$

Hence, the convolution (2.2) belongs to $L(R_+)$. Furthermore,

$$(F_s f)(x)\,(F_s g)(x) = \frac{2}{\pi} \int_0^\infty \int_0^\infty \sin xu \sin xv\, f(u)\, g(v)\, du\, dv$$

$$= \frac{1}{\pi} \int_0^\infty \int_0^\infty \cos x(u-v)\, f(u)g(v)du\, dv - \frac{1}{\pi} \int_0^\infty \int_0^\infty \cos x(u+v)\, f(u)g(v)du\, dv$$

$$= \frac{1}{\pi} \int_0^\infty \int_0^\infty \cos xt\,[f(y)g(y+t) + f(y+t)g(y)]dy\, dt$$

$$-\frac{1}{\pi} \int_0^\infty \int_0^t \cos xt\, f(y)g(t-y)dy\, dt$$

$$= \frac{1}{\pi} \int_0^\infty \cos xt\ \left[\int_0^\infty f(y)g(y+t)dy + \int_t^\infty f(y)g(y-t)dy\ - \int_0^t f(y)g(t-y)dy\right] dt$$

$$= \frac{1}{\pi} \int_0^\infty \cos xt\ \left[\int_0^\infty f(y)g(y+t)dy + \int_0^\infty \mathrm{sign}(y-t) f(y) g(|y-t|)dy\ \right] dt$$

$$= F_c(f \overset{2}{*} g)(x).$$

Theorem 1 is thus proved.

Remark 1: Formulas (1.4) and (2.2) show that the convolution $f \overset{0}{*} g$ and $f \overset{2}{*} g$ are commutative. On the other hand, the convolution $f \overset{1}{*} g$ is non commutative:

$$f \overset{1}{*} g = -g \overset{1}{*} f + \sqrt{\frac{2}{\pi}} f \underset{L}{*} g, \tag{2.3}$$

where $f \underset{L}{*} g$ is the Laplace convolution

$$(f \underset{L}{*} g)(x) = \int_0^x f(y)\, g(x-y)\, dy. \tag{2.4}$$

Indeed, we have

$$(f \overset{1}{*} g)(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(y)[g(\mid x-y \mid) - g(x+y)]dy$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \int_{-x}^\infty f(x+s) g(\mid s \mid)\, ds - \int_x^\infty f(s-x) g(s)\, ds \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \int_0^\infty g(s)[f(x+s) - f(\mid s-x \mid)]\, ds + \right.$$

$$\left. + \int_{-x}^0 f(x+s)\, g(\mid s \mid)\, ds + \int_0^x f(\mid s-x \mid)\, g(s)\, ds \right\}$$

$$= -(g \overset{1}{*} f)(x) + \sqrt{\frac{2}{\pi}} (f \underset{L}{*} g)(x).$$

Remark 2: Convolution (2.1) was introduced implicitly, but incorrectly in [8], where the term $\mathrm{sign}(y-x)$ was missing.

Theorem 2. *Let the functions f, g, h belong to $\in L(R_+)$. Then the following formulas hold*

$$(f \overset{1}{*} g) \overset{1}{*} h = (f \overset{1}{*} h) \overset{1}{*} g = f \overset{1}{*} (g \overset{0}{*} h), \tag{2.5}$$

$$f \overset{0}{*} (g \overset{2}{*} h) = g \overset{2}{*} (h \overset{1}{*} f) = h \overset{2}{*} (g \overset{1}{*} f), \tag{2.6}$$

$$f \overset{1}{*} (g \overset{2}{*} h) = g \overset{1}{*} (f \overset{2}{*} h) = h \overset{1}{*} (f \overset{2}{*} g), \tag{2.7}$$

$$f \overset{0}{*} (g \overset{0}{*} h) = g \overset{0}{*} (f \overset{0}{*} h) = h \overset{0}{*} (f \overset{0}{*} g), \tag{2.8}$$

The proof follows easily from formulas (1.4), (1.6) and (2.2). For example, we have

$$F_s[(f \overset{1}{*} g) \overset{1}{*} h] = F_s(f \overset{1}{*} g) F_c(h) = F_s(f) F_c(g) F_c(h)$$

$$= [F_s(f) F_c(h)] F_c(g) = F_s(f \overset{1}{*} h) F_c(g)$$

$$= F_s[(f \overset{1}{*} h) \overset{1}{*} g].$$

Hence, $(f \overset{1}{*} g) \overset{1}{*} h = (f \overset{1}{*} h) \overset{1}{*} g$. On the other hand,

$$F_s[(f \overset{1}{*} g) \overset{1}{*} h] = F_s(f)[F_c(g)F_c(h)]$$

$$= F_s(f)F_c(g \overset{0}{*} h) = F_s[(f \overset{1}{*} (g \overset{0}{*} h)].$$

Therefore, $(f \overset{1}{*} g) \overset{1}{*} h = f \overset{1}{*} (g \overset{0}{*} h)$, and formula (2.5) is proved. By the same way, one can verify the other parts, too.

3. Applications to integral equations

We consider the following linear system of integral equations:

$$\varphi(x) + \lambda_1 \int_0^\infty k(x,y)\psi(y)dy = f(x), \tag{3.1}$$

$$\psi(x) + \lambda_2 \int_0^\infty h(x,y)\varphi(y)dy = g(x),\ x \in R_+ \ . \tag{3.2}$$

where φ and ψ are unknown functions, f and g are given functions, λ_1 and λ_2 denote complex parameters, and $k(x,y)$ and $h(x,y)$ are the kernels that can be expressed in the form

$$k(x,y) = k_1(\mid x-y \mid) - k_1(x+y),$$

$$h(x,y) = \text{sign}(y-x)h_1(\mid x-y \mid) + h_1(x+y). \tag{3.3}$$

Applying the Fourier sine transform to equation (3.1) and the Fourier cosine transform to equation (3.2) and using the convolution formulas (1.6) and (2.2) we obtain a linear system of algebraic equations

$$F_s(\varphi) + \sqrt{2\pi}\lambda_1 F_c(\psi)F_s(k_1) = F_s(f),$$

$$F_c(\psi) + \sqrt{2\pi}\lambda_2 F_s(\varphi)F_s(h_1) = F_c(g). \tag{3.4}$$

Suppose that

$$1 - 2\pi\lambda_1\lambda_2(F_s k_1)(x)(F_s h_1)(x) \neq 0 \tag{3.5}$$

for any $x \in R_+$. Then the linear system (3.4) has the solution

$$F_s(\varphi) = [F_s(f) - \sqrt{2\pi}\lambda_1 F_c(g)F_s(k_1)]/[1 - 2\pi\lambda_1\lambda_2 F_s(k_1)F_s(h_1)],$$

$$F_c(\psi) = [F_c(g) - \sqrt{2\pi}\lambda_2 F_s(f)F_s(h_1)]/[1 - 2\pi\lambda_1\lambda_2 F_s(k_1)F_s(h_1)]. \quad (3.6)$$

Consider the function $\nu(t) = 2\pi\lambda_1\lambda_2 t/(1 - 2\pi\lambda_1\lambda_2 t)$ with $t = F_c(k_1 \overset{2}{*} h_1) = F_s(k_1)F_s(h_1)$. Since $\nu(t)$ is analytic under the condition (3.5) and $\nu(0) = 0$, by the Wiener-Levi theorem there exists a function $l \in L(R_+)$ such that

$$F_c(l) = 2\pi\lambda_1\lambda_2 F_s(k_1)F_s(h_1)/[1 - 2\pi\lambda_1\lambda_2 F_s(k_1)F_s(h_1)]. \quad (3.7)$$

Hence, we obtain

$$F_s(\varphi) = [F_s(f) - \sqrt{2\pi}\lambda_1 F_c(g)F_s(k_1)][1 + F_c(l)]$$

$$= F_s(f) - \sqrt{2\pi}\lambda_1 F_s(k_1 \overset{1}{*} g) + F_s(f \overset{1}{*} l) - \sqrt{2\pi}\lambda_1 F_s((k_1 \overset{1}{*} g) \overset{1}{*} l).$$

Therefore,

$$\varphi = f - \sqrt{2\pi}\lambda_1 k_1 \overset{1}{*} g + f \overset{1}{*} l - \sqrt{2\pi}\lambda_1 (k_1 \overset{1}{*} g) \overset{1}{*} l \ . \quad (3.8)$$

Similarly, we have

$$F_c(\psi) = [F_c(g) - \sqrt{2\pi}\lambda_2 F_s(f)F_s(h_1)][1 + F_c(l)]$$

$$= F_c(g) - \sqrt{2\pi}\lambda_2 F_c(f \overset{2}{*} h_1) + F_c(l \overset{0}{*} g) - \sqrt{2\pi}\lambda_2 F_c(l \overset{0}{*} (f \overset{2}{*} h_1)).$$

Consequently,

$$\psi = g - \sqrt{2\pi}\lambda_2 f \overset{2}{*} h_1 + l \overset{0}{*} g - \sqrt{2\pi}\lambda_2 l \overset{0}{*} (f \overset{2}{*} h_1). \quad (3.9)$$

References

1. Churchill R.V. *Fourier series and boundary value problems.* New York, 1941.

2. Glaeske H.-J. and Vu Kim Tuan. Convolution of the Hilbert transform and its application to some nonlinear singular integral equations. *Integral Transforms and Special Functions* **3**(1995), no. 4, p. 263-268.

3. Kakichev V.A. On the convolutions for integral transform (Russian).

Vestsi Akad. Navuk BSSR, Ser. Fiz. Mat. Navuk (1967), no. 2, p. 48-57.
4. Kakichev V.A. On the matrix convolutions for power series (Russian). *Izv. Vyssh. Uchebn. Zaved. Mat.* (1990), no. 2, p. 53-62.
5. Kakichev V.A. and Nguyen Xuan Thao. On the design method for the generalized integral convolutions (Russian). *Izv. Vyssh. Uchebn. Zaved. Mat.* (1997), no. 6.
6. Nguyen Thanh Hai and Yakubovich S.B. *The double Mellin-Barnes type integrals and their applications to convolution theory.* World Scientific, Singapore, 1992.
7. Saigo M. and Yakubovich S.B. On the theory of convolution integrals for G-transforms. *Fukuoka Univ. Sci. Reports* **21**(1991), no. 2, p. 181-193.
8. Sneddon I.N. *The use of integral transforms.* McGray–Hill, New York, 1972.
9. Srivastava H.M. and Vu Kim Tuan. A new convolution theorem for the Stieltjes transform and its application to a class of singular integral equations. *Arch. Math.* **64**(1995), no. 2, p. 144-149.
10. Vu Kim Tuan and Megumi Saigo. Convolution of Hankel transform and its application to an integral involving Bessel function of first kind. *Internat. J. Math. Math. Sci.* **18**(1995), no. 3, p. 545-550.
11. Yakubovich S.B. On the constructive method of integral convolutions (Russian). *Dokl. Akad. Nauk BSSR* **34**(1990), no. 7, p. 588-591.
12. Yakubovich S.B. and Luchko Yu.F. *The hypergeometric approach to integral transforms and convolutions.* Kluwer Acad. Publ., 1994.