ON THE GENERALIZED CONVOLUTIONS
FOR FOURIER COSINE AND SINE TRANSFORMS

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Abstract. A generalized convolution for the Fourier cosine and sine transforms is introduced, its properties and applications to integral equations are considered.

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1. Introduction

Generalized convolution of functions $f$ and $g$ under three operators $K$, $K_1$, $K_2$, and with some weight-function $\gamma$ is a function, denoted by the symbol $f * g$, such that the following factorization property holds [5]:

$$K(f * g)(x) = \gamma(x)(K_1f)(x)(K_2g)(x).$$

(1.1)

If $K = K_1 = K_2$ we have the usual classical convolution [3,4]. For example, for $K = K_1 = K_2 = F_c$ – the Fourier cosine transform [8]

$$ (F_c f)(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(y) \cos xy \, dy, $$

(1.2)
the convolution has the form [8]

\[(f \ast g)(x) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} f(y)[g(|x-y|) + g(x+y)]dy, \quad (1.3)\]

and the property (1.1) holds

\[F_c(f \ast g)(x) = (F_c f)(x) (F_c g)(x). \quad (1.4)\]

Otherwise, there appear "exotic" convolutions. An example of generalized convolutions was first introduced by Churchill [1]

\[(f \ast g)(x) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} f(y)[g(|x-y|) - g(x+y)]dy, \quad (1.5)\]

and the respective factorization property (1.1) for (1.5) has the form

\[F_s(f \ast g)(x) = (F_s f)(x) (F_c g)(x), \quad (1.6)\]

where \(F_s\) is the Fourier sine transform [8]

\[(F_s f)(x) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(y) \sin xy dy. \quad (1.7)\]

Many authors have been studied similar convolutions for Hankel’s transform [10], Stieltjes’ transform [9], Hilbert’s transform [2], G-transform [7], and integral transforms of Mellin convolution type [6,12]. The present work is devoted to investigate properties of another generalized convolution for Fourier cosine and sine transforms, different from (1.5), and its application to a linear system of integral equations.

2. The generalized convolution

**Definition 1.** A generalized convolution for the Fourier cosine and sine transforms is defined as follows:

\[(f \ast g)(x) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} f(y)[\text{sign}(y-x)g(|y-x|) + g(y+x)]dy. \quad (2.1)\]
**Theorem 1.** Let \( f, g \in L(R_+) \), then the convolution \( f \ast g \) belongs to \( L(R_+) \) and

\[
F_c(f \ast g)(x) = (F_s f)(x)(F_s g)(x), \quad x \in R_+. \tag{2.2}
\]

**Proof.** We have

\[
\int_0^\infty |(f \ast g)(x)| \, dx \leq \frac{1}{\sqrt{2\pi}} \int_0^\infty \int_0^\infty |(f(y))| \, dx + \int_0^\infty |g(x)| \, dx
\]

\[
\leq \frac{1}{\sqrt{2\pi}} \int_0^\infty |f(y)| \, dy \left[ \int_0^\infty |g(y)| \, dy + \int_0^\infty |g(x)| \, dx \right] = \sqrt{\frac{2}{\pi}} \int_0^\infty |f(y)| \, dy \int_0^\infty |g(x)| \, dx < \infty.
\]

Hence, the convolution (2.2) belongs to \( L(R_+) \). Furthermore,

\[
(F_s f)(x)(F_s g)(x) = \frac{2}{\pi} \int_0^\infty \int_0^\infty \sin xu \sin xv f(u)g(v) \, du \, dv
\]

\[
= \frac{1}{\pi} \int_0^\infty \int_0^\infty \cos x(u-v) f(u)g(v) \, du \, dv - \frac{1}{\pi} \int_0^\infty \int_0^\infty \cos x(u+v) f(u)g(v) \, du \, dv
\]

\[
= \frac{1}{\pi} \int_0^\infty \int_0^\infty \cos x t [f(y)g(y+t) + f(y+t)g(y)]dy \, dt - \frac{1}{\pi} \int_0^\infty \int_0^t \cos x t f(y)g(t-y)dy \, dt
\]

\[
= \frac{1}{\pi} \int_0^\infty \cos x t \left[ \int_0^\infty f(y)g(y+t)dy + \int_t^\infty f(y)g(y-t)dy - \int_0^t f(y)g(t-y)dy \right] dt
\]

\[
= \frac{1}{\pi} \int_0^\infty \cos x t \left[ \int_0^\infty f(y)g(y+t)dy + \int_0^\infty \text{sign}(y-t)f(y)g(|y-t|)dy \right] dt
\]

\[
F_c(f \ast g)(x) = F_c(f \ast g)(x).
\]

Theorem 1 is thus proved.

**Remark 1:** Formulas (1.4) and (2.2) show that the convolution \( f \ast g \) and \( f \ast g \) are commutative. On the other hand, the convolution \( f \ast g \) is non-commutative:

\[
f \ast g = -g \ast f + \sqrt{\frac{2}{\pi}} f \ast L g, \tag{2.3}
\]

4
where $f^*_Lg$ is the Laplace convolution

$$ (f^*_Lg)(x) = \int_0^x f(y) g(x - y) \, dy. \quad (2.4) $$

Indeed, we have

$$ (f \ast g)(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(y)[g(|x - y|) - g(x + y)] \, dy $$

$$ = \frac{1}{\sqrt{2\pi}} \left\{ \int_{-x}^\infty f(x + s) g(|s|) \, ds - \int_x^\infty f(s - x) g(s) \, ds \right\} $$

$$ = \frac{1}{\sqrt{2\pi}} \left\{ \int_0^\infty g(s)[f(x + s) - f(|s - x|)] \, ds + \int_{-x}^0 f(x + s) g(|s|) \, ds + \int_x^\infty f(|s - x|) g(s) \, ds \right\} $$

$$ = - (g \ast f)(x) + \sqrt{\frac{2}{\pi}} (f^*_Lg)(x). $$

**Remark 2:** Convolution (2.1) was introduced implicitly, but incorrectly in [8], where the term sign($y - x$) was missing.

**Theorem 2.** Let the functions $f, g, h$ belong to $\in L(R_+)$. Then the following formulas hold

\[
(f \ast g) \ast h = (f \ast h) \ast g = f \ast (g \ast h), \tag{2.5}
\]

\[
f^0 (g \ast h) = g^2 (h \ast f) = h^2 (g \ast f), \tag{2.6}
\]

\[
f^1 (g \ast h) = g^1 (f \ast h) = h^1 (f \ast g), \tag{2.7}
\]

\[
f^0 (g \ast h) = g^0 (f \ast h) = h^0 (f \ast g). \tag{2.8}
\]

The proof follows easily from formulas (1.4), (1.6) and (2.2). For example, we have

$$ F_s[(f^\frac{1}{2} g)^\frac{1}{2} h] = F_s(f^\frac{1}{2} g)F_c(h) = F_s(f)F_c(g)F_c(h) $$

$$ = [F_s(f)F_c(h)]F_c(g) = F_s(f^\frac{1}{2} h)F_c(g) $$
Hence, \((f \ast h) \ast g = (f \ast g) \ast h\). On the other hand,
\[
F_s[(f \ast g) \ast h] = F_s(f)F_c(g)F_c(h) = F_s((f \ast g) \ast h).
\]
Therefore, \((f \ast g) \ast h = f \ast (g \ast h)\), and formula (2.5) is proved. By the same way, one can verify the other parts, too.

3. Applications to integral equations

We consider the following linear system of integral equations:

\[
\begin{align*}
\varphi(x) + \lambda_1 \int_0^\infty k(x, y)\psi(y)dy &= f(x), \quad (3.1) \\
\psi(x) + \lambda_2 \int_0^\infty h(x, y)\varphi(y)dy &= g(x), \quad x \in R_+.
\end{align*}
\]

where \(\varphi\) and \(\psi\) are unknown functions, \(f\) and \(g\) are given functions, \(\lambda_1\) and \(\lambda_2\) denote complex parameters, and \(k(x, y)\) and \(h(x, y)\) are the kernels that can be expressed in the form

\[
\begin{align*}
k(x, y) &= k_1(|x - y|) - k_1(x + y), \\
h(x, y) &= \text{sign}(y - x)h_1(|x - y|) + h_1(x + y).
\end{align*}
\]

Applying the Fourier sine transform to equation (3.1) and the Fourier cosine transform to equation (3.2) and using the convolution formulas (1.6) and (2.2) we obtain a linear system of algebraic equations

\[
\begin{align*}
F_s(\varphi) + \sqrt{2\pi} \lambda_1 F_c(\psi)F_s(k_1) &= F_s(f), \\
F_c(\psi) + \sqrt{2\pi} \lambda_2 F_s(\varphi)F_s(h_1) &= F_c(g).
\end{align*}
\]

Suppose that

\[
1 - 2\pi \lambda_1 \lambda_2 (F_s(k_1)(x)(F_s(h_1))(x) \neq 0
\]

for any \(x \in R_+\). Then the linear system (3.4) has the solution

\[
F_s(\varphi) = [F_s(f) - \sqrt{2\pi} \lambda_1 F_c(g)F_s(k_1)]/[1 - 2\pi \lambda_1 \lambda_2 F_s(k_1)F_s(h_1)],
\]
\[ F_c(\psi) = \left[ F_c(g) - \sqrt{2\pi\lambda_2} F_s(f) F_s(h_1) \right] / \left[ 1 - 2\pi\lambda_1\lambda_2 F_s(k_1) F_s(h_1) \right]. \]  

(3.6)

Consider the function \( \nu(t) = 2\pi\lambda_1\lambda_2 t / (1 - 2\pi\lambda_1\lambda_2 t) \) with \( t = F_c(k_1 \ast h_1) = F_s(k_1) F_s(h_1) \). Since \( \nu(t) \) is analytic under the condition (3.5) and \( \nu(0) = 0 \), by the Wiener-Levi theorem there exists a function \( l \in L(R_+) \) such that

\[ F_c(l) = 2\pi\lambda_1\lambda_2 F_s(k_1) F_s(h_1) / [1 - 2\pi\lambda_1\lambda_2 F_s(k_1) F_s(h_1)]. \]  

(3.7)

Hence, we obtain

\[ F_s(\varphi) = \left[ F_s(f) - \sqrt{2\pi\lambda_1} F_c(g) F_s(k_1) \right] [1 + F_c(l)] \]

\[ = F_s(f) - \sqrt{2\pi\lambda_1} F_s(k_1 \ast g) + F_s(f \ast l) - \sqrt{2\pi\lambda_1} F_s((k_1 \ast g) \ast l). \]

Therefore,

\[ \varphi = f - \sqrt{2\pi\lambda_1} k_1 \ast g + f \ast l - \sqrt{2\pi\lambda_1} (k_1 \ast g) \ast l. \]  

(3.8)

Similarly, we have

\[ F_c(\psi) = \left[ F_c(g) - \sqrt{2\pi\lambda_2} F_s(f) F_s(h_1) \right] / \left[ 1 - 2\pi\lambda_1\lambda_2 F_s(k_1) F_s(h_1) \right] \]

\[ = F_c(g) - \sqrt{2\pi\lambda_2} F_c(f \ast h_1) + F_c(l \ast g) - \sqrt{2\pi\lambda_2} F_c(l \ast (f \ast h_1)). \]

Consequently,

\[ \psi = g - \sqrt{2\pi\lambda_2} f \ast h_1 + l \ast g - \sqrt{2\pi\lambda_2} (l \ast (f \ast h_1)). \]  

(3.9)

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