

MODIFIED FRACTIONAL INTEGRAL OPERATORS IN L_p SPACE WITH POWER-LOGARITHMIC WEIGHT

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Abstract

The modified fractional integral operators, introduced in [4], are proved to be bounded from some L_{p^-} space into some L_{q^-} space with power-logarithmic weights.

1. Introduction

In [4] the multidimensional modified fractional integrals of order $\alpha > 0$ are defined as follows

$$X_{+,n}^\alpha f(x) = \frac{1}{\Gamma(\alpha+1)} \frac{\partial^n}{\partial x_1 \dots \partial x_n} \int_{R_+^n} \left[\min \left\{ \frac{x_1}{t_1}, \dots, \frac{x_n}{t_n} \right\} - 1 \right]_+^\alpha f(t) dt, \quad (1)$$

$$X_{-,n}^\alpha f(x) = \frac{(-1)^n}{\Gamma(\alpha+1)} \frac{\partial^n}{\partial x_1 \dots \partial x_n} \int_{R_+^n} \left[1 - \max \left\{ \frac{x_1}{t_1}, \dots, \frac{x_n}{t_n} \right\} \right]_+^\alpha f(t) dt. \quad (2)$$

Here $R_+^n = \{(t_1, \dots, t_n) \mid t_1, \dots, t_n > 0\}$ and $\varphi_+(x)$ is constructed from a real-valued function $\varphi(x)$ by

$$\varphi_+(x) = \begin{cases} \varphi(x) & , \quad \varphi(x) > 0 \\ 0 & , \quad \varphi(x) \leq 0 \end{cases}. \quad (3)$$

Operators $X_{+,n}^\alpha$, $X_{-,n}^\alpha$ generalize the one-dimensional Riemann-Liouville and Weyl fractional integral operators [2], respectively. Some of their properties (product rules, index laws, mapping properties, composition structures, inverses, ...) are established in the

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space of functions $\mathcal{M}_\gamma(R_+^n)$ [5].

In this paper the modified fractional integral operators $X_{+,n}^\alpha$, $X_{-,n}^\alpha$ on $\mathcal{M}_\gamma(R_+^n)$ are proved to satisfy some $L_p - L_q$ weighted inequalities. Hence, they can be continuously extended to some bounded operators in the classical space L_p with power-logarithmic weight. Many properties of $X_{+,n}^\alpha$ and $X_{-,n}^\alpha$ established in [4] only on $\mathcal{M}_\gamma(R_+^n)$, therefore, are still valid in the space L_p with weight.

2. Pitt's Inequality for Mellin Transform

Let $\hat{f}(\xi)$ be the Fourier transform of $f(x) : R^n \rightarrow C$

$$\hat{f}(\xi) = \int_{R^n} f(x)e^{-ix\xi}dx. \quad (4)$$

The following Pitt's inequality holds

$$\|\|\xi\|^{-b}\hat{f}(\xi)\|_r \leq C\| |x|^a f(x) \|_p \quad (5)$$

if

$$0 \leq b < \frac{n}{r}, \quad 0 \leq a = b + n \left(1 - \frac{1}{p} - \frac{1}{r}\right), \quad 1 < p \leq r < \infty, \quad (6)$$

(see [3]). Inequality (5) under condition (6) is still valid after a linear transformation

$$\|\|\xi - \tau\|^{-b}\hat{f}(\xi)\|_r \leq C\| |x - t|^a f(x) \|_p, \quad \tau, t \in R^n. \quad (7)$$

Consider now the Mellin transform

$$f^*(s) = \int_{R_+^n} f(t)t^{s-1}dt, \quad \Re(s) = \gamma, \quad \gamma \in R^n, \quad (8)$$

and its inversion

$$f(t) = \frac{1}{(2\pi i)^n} \int_{(\gamma)} f^*(s)t^{-s}ds. \quad (9)$$

Here (γ) stands for $(\gamma_1 - i\infty, \gamma_1 + i\infty) \times \cdots \times (\gamma_n - i\infty, \gamma_n + i\infty)$. After changing $t = e^x$ and $s = \gamma + iu$ we obtain

$$f^*(\gamma + iu) = \int_{R^n} e^{\gamma x} f(e^x) e^{ixu} dx, \quad (10)$$

$$e^{\gamma x} f(e^x) = \frac{1}{(2\pi)^n} \int_{R^n} f^*(\gamma + iu) e^{-ixu} du. \quad (11)$$

Applying the Pitt's inequality to the Fourier transforms (10) and (11) we get

$$\|\|u - \tau\|^{-b} f^*(\gamma + iu)\|_r \leq C\| |x - t|^a e^{\gamma x} f(e^x) \|_p, \quad (12)$$

$$\|\|x - t\|^{-b} e^{\gamma x} f(e^x)\|_r \leq C\| |u - \tau|^a f^*(\gamma + iu) \|_p, \quad \tau, t \in R^n, \quad (13)$$

if condition (6) is satisfied. Changing back $x = \ln y$ and $\gamma + iu = s$ we get

$$\| |s - \gamma - i\tau|^{-b} f^*(s) \|_{L_r((\gamma))} \leq C \| |\ln y - t|^a y^{\gamma - \frac{1}{p}} f(y) \|_{L_p(R_+^n)}, \quad (14)$$

$$\| |\ln y - t|^{-b} y^{\gamma - \frac{1}{r}} f(y) \|_{L_r(R_+^n)} \leq C \| |s - \gamma - i\tau|^a f^*(s) \|_{L_p((\gamma))}, \quad \tau, t \in R^n. \quad (15)$$

It is the Pitt's inequalities for the Mellin transforms (8) and (9). Because $a, b \geq 0$, then $|s|^{-b} \leq C |s - \gamma - i\tau|^{-b}$, $|s - \gamma - i\tau|^a \leq C |s|^a$, $(1 + |\ln y|)^{-b} \leq C |\ln y - t|^{-b}$, $|\ln y - t|^a \leq C(1 + |\ln y|)^a$, and therefore we have

$$\| |s|^{-b} f^*(s) \|_{L_r((\gamma))} \leq C \| (1 + |\ln y|)^a y^{\gamma - \frac{1}{p}} f(y) \|_{L_p(R_+^n)}, \quad (16)$$

$$\| (1 + |\ln y|)^{-b} y^{\gamma - \frac{1}{r}} f(y) \|_{L_r(R_+^n)} \leq C \| |s|^a f^*(s) \|_{L_p((\gamma))}, \quad \tau, t \in R^n. \quad (17)$$

3. Weighted Inequalities for Convolution Operators

Consider the following convolution integral operator

$$(Kf)(x) = \int_{R_+^n} k(xy) f(y) dy, \quad (18)$$

where the kernel $k(x)$ satisfies the condition that the integral

$$\int_{\epsilon}^E k(y) y^{s-1} dy, \quad \Re(s) = \frac{1}{2}, \quad 0 < \epsilon_j < E_j < \infty, \quad j = 1, 2, \dots, n, \quad (19)$$

is uniformly bounded with respect to $\epsilon, E \in R_+^n$, $\Re(s) = 1/2$, and that the integral (19) converges to some function $k^*(s)$ as $\epsilon \rightarrow 0, E \rightarrow \infty$. It is easy to see that if f is represented by (9) for some $f^*(s) \in L_1((1/2))$, then integral (18) converges in the meaning thus explained above, and improper and

$$\int_{R_+^n} k(xy) f(y) dy = \frac{1}{(2\pi i)^n} \int_{(1/2)} k^*(s) f^*(1-s) x^{-s} ds, \quad (20)$$

(see [5] for one-dimensional case). Hence, the Mellin transform of $(Kf)(x)$ is $k^*(s) f^*(1-s)$. From the condition imposed on $k(x)$ we see that $k^*(s)$ is bounded on $(1/2)$. Suppose additionally that

$$|k^*(s)| \leq C |s|^{-\alpha}, \quad \alpha \geq 0, \quad s \in (1/2). \quad (21)$$

Using inequality (15) for $(Kf)(x)$ we have

$$\begin{aligned} \| |\ln y - t|^{-b} y^{\frac{1}{2} - \frac{1}{r}} (Kf)(y) \|_{L_r(R_+^n)} &\leq C \| |s - \gamma - i\tau|^a k^*(s) f^*(1-s) \|_{L_p((1/2))} \\ &\leq C \| |s - \gamma - i\tau|^{a-\alpha} f^*(s) \|_{L_p((1/2))}, \quad \tau, t \in R^n \end{aligned} \quad (22)$$

where a, b, p, r satisfy condition (6). Suppose now $a \leq \alpha$. Applying inequality (14) we get

$$\| |s - \gamma - i\tau|^{a-\alpha} f^*(s) \|_{L_p((1/2))} \leq C \| |\ln y - t|^d y^{\frac{1}{2} - \frac{1}{q}} f(y) \|_{L_q(R_+^n)}, \quad \tau, t \in R^n, \quad (23)$$

if

$$0 \leq \alpha - a < \frac{n}{p}, \quad 0 \leq d = \alpha - a + n\left(1 - \frac{1}{p} - \frac{1}{q}\right), \quad 1 < q \leq p. \quad (24)$$

Combining (23) with (22) we obtain

$$\| |\ln y - t|^{-b} y^{\frac{1}{2} - \frac{1}{r}} (Kf)(y) \|_{L_r(R_+^n)} \leq C \| |\ln y - t|^d y^{\frac{1}{2} - \frac{1}{q}} f(y) \|_{L_q(R_+^n)}, \quad \tau, t \in R^n, \quad (25)$$

or, equivalently,

$$\| |\ln y - t|^{-b} y^{\frac{1}{2} - \frac{1}{r}} (Kf)(y) \|_{L_r(R_+^n)} \leq C \| |\ln y - t|^{\alpha - b + n(\frac{1}{r} - \frac{1}{q})} y^{\frac{1}{2} - \frac{1}{q}} f(y) \|_{L_q(R_+^n)}, \quad \tau, t \in R^n, \quad (26)$$

provided that

$$\begin{aligned} 0 \leq b < \frac{n}{r}, \quad \alpha + n\left(\frac{1}{r} - 1\right) < b \leq \alpha + n\left(\frac{1}{p} + \frac{1}{r} - 1\right), \\ b + n\left(1 - \frac{1}{p} - \frac{1}{r}\right), \quad \alpha - b + n\left(\frac{1}{r} - \frac{1}{q}\right) \geq 0, \quad 1 < q \leq p \leq r < \infty. \end{aligned} \quad (27)$$

In particular,

$$\| (|\ln y| + 1)^{-b} y^{\frac{1}{2} - \frac{1}{r}} (Kf)(y) \|_{L_r(R_+^n)} \leq C \| (|\ln y| + 1)^{\alpha - b + n(\frac{1}{r} - \frac{1}{q})} y^{\frac{1}{2} - \frac{1}{q}} f(y) \|_{L_q(R_+^n)}. \quad (28)$$

4. Weighted Inequalities for Modified Fractional Operators

It is proved [4] that if

$$\Re(\beta_j) < 1/2, \quad j = 1, \dots, n, \quad \Re(\alpha) + \sum_{j=1}^n \Re(\beta_j) < n/2, \quad (29)$$

then the operator $x^\beta X_{+,n}^\alpha x^{-\beta}$ is a homeomorphism of $\mathcal{M}_{1/2}(R_+^n)$ and moreover,

$$x^\beta X_{+,n}^\alpha x^{-\beta} = \frac{1}{(2\pi i)^n} \int_{(1/2)} \frac{\Gamma\left(n - \alpha - \sum_{j=1}^n \beta_j - \sum_{j=1}^n s_j\right)}{\Gamma\left(n - \sum_{j=1}^n \beta_j - \sum_{j=1}^n s_j\right)} f^*(s) x^{-s} ds. \quad (30)$$

Since $\frac{\Gamma\left(n - \alpha - \sum_{j=1}^n \beta_j - \sum_{j=1}^n s_j\right)}{\Gamma\left(n - \sum_{j=1}^n \beta_j - \sum_{j=1}^n s_j\right)} = O(|s|^{-\alpha})$, one can apply inequality (26) to obtain

$$\| |\ln y - t|^{-b} y^{\frac{1}{2} - \frac{1}{r} + \beta} X_{+,n}^\alpha y^{-\beta} f(y) \|_{L_r(R_+^n)} \leq C \| |\ln y - t|^{\alpha - b + n(\frac{1}{r} - \frac{1}{q})} y^{\frac{1}{2} - \frac{1}{q}} f(y) \|_{L_q(R_+^n)}, \quad \tau, t \in R^n, \quad (31)$$

provided that the conditions (27) and (29) are satisfied.

Similarly, if

$$\Re(\beta_j) > 1/2, \quad j = 1, \dots, n, \quad (32)$$

then the operator $x^\beta X_{-,n}^\alpha x^{-\beta}$ is a homeomorphism of $\mathcal{M}_{1/2}(R_+^n)$ and moreover,

$$x^\beta X_{-,n}^\alpha x^{-\beta} = \frac{1}{(2\pi i)^n} \int_{(1/2)} \frac{\Gamma(1 - n + \sum_{j=1}^n \beta_j + \sum_{j=1}^n s_j)}{\Gamma(1 + \alpha - n + \sum_{j=1}^n \beta_j + \sum_{j=1}^n s_j)} f^*(s) x^{-s} ds. \quad (33)$$

Since $\frac{\Gamma(1 - n + \sum_{j=1}^n \beta_j + \sum_{j=1}^n s_j)}{\Gamma(1 + \alpha - n + \sum_{j=1}^n \beta_j + \sum_{j=1}^n s_j)} = O(|s|^{-\alpha})$, one can apply inequality (26) again to obtain

$$\| |\ln y - t|^{-b} y^{\frac{1}{2} - \frac{1}{r} + \beta} X_{-,n}^\alpha y^{-\beta} f(y) \|_{L_r(R_+^n)} \leq C \| |\ln y - t|^{\alpha - b + n(\frac{1}{r} - \frac{1}{q})} y^{\frac{1}{2} - \frac{1}{q}} f(y) \|_{L_q(R_+^n)}, \quad \tau, t \in R^n, \quad (34)$$

provided that the conditions (27) and (32) are satisfied.

Function f belongs to $\mathcal{M}_{1/2}(R_+^n)$ if and only if f has the representation (9), where $f^*(s)$ is infinitely differentiable and has a compact support on $(1/2)$. The set of these functions $f^*(s)$ is dense in $L_2((1/2))$, therefore, by the Plancherel theorem for the Mellin transform [1], the set $\mathcal{M}_{1/2}(R_+^n)$ is dense in $L_2(R_+^n)$, and hence, dense in any L_p - space with local integrable weight. Consequently, from (31) and (34) one can see that $x^\beta X_{+,n}^\alpha x^{-\beta}$ and $x^\beta X_{-,n}^\alpha x^{-\beta}$ can be extended from $\mathcal{M}_{1/2}(R_+^n)$ to bounded operators from L_p with weight $|\ln y - t|^{-b} y^{\frac{1}{2} - \frac{1}{r} + \beta}$ to L_q with weight $|\ln y - t|^{\alpha - b + n(\frac{1}{r} - \frac{1}{q})} y^{\frac{1}{2} - \frac{1}{q}}$, where the parameters satisfy conditions (27) and either (29) or (32), respectively.

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