

COMPOSITION OF ERDÉLYI-KOBER FRACTIONAL OPERATORS

Leda Galué, S. L. Kalla, and Vu Kim Tuan

Centro de Investigación de Matemática Aplicada (C.I.M.A.)
Facultad de Ingeniería, La Universidad del Zulia
Apartado de Correo 10.482
Maracaibo, Venezuela

and

Department of Mathematics and Computer Science
Faculty of Science, Kuwait University
P.O. Box 5969, Safat 13060, Kuwait

Abstract

In this paper we consider the Erdélyi-Kober fractional differentiation operator ${}_aI_x^{\eta,\alpha}$ with nonzero lower limit of integration; and express it through an integral operator including the Gauss hypergeometric function in the kernel. The composition ${}_bI_x^{\gamma,\beta} {}_aI_x^{\eta,\alpha}$ of Erdélyi-Kober operators with different lower limits of integration is studied for the cases $Re(\alpha), Re(\beta) > 0$, and $Re(\alpha) \leq 0 < Re(\beta)$.

Key words: Fractional integration and differentiation, composition of operators.

1. Introduction

The concept of differ-integral of complex order ν , which is a generalization of the ordinary n -th derivative and n times integral to any complex number ν [13], can be introduced in several ways. The most widely used definition of an integral of fractional order is via an integral transform, called the Riemann-Liouville operator of fractional integration [16,19]

$$\begin{aligned} {}_a I_x^\alpha f(x) &= \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \quad \operatorname{Re}(\alpha) > 0, \\ &= \frac{d^n}{dx^n} {}_a I_x^{\alpha+n} f(x), \quad -n < \operatorname{Re}(\alpha) \leq 0. \end{aligned} \quad (1)$$

Several authors ([1], [3]-[7], [9], [10], [18], [19]) have defined and studied operators of fractional integration via integral transform approach.

An operator of fractional integration generalizing (1) through association of power weight was introduced by Erdélyi and Kober (1940), Kober (1940), Erdélyi (1951)

$$\begin{aligned} {}_a I_x^{\eta,\alpha} f(x) &= \frac{x^{-\eta-\alpha}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} t^\eta f(t) dt, \quad \operatorname{Re}(\alpha) > 0, \\ &= x^{-\eta-\alpha} \frac{d^n}{dx^n} x^{\eta+\alpha} {}_a I_x^{\eta,\alpha+n} f(x), \quad -n < \operatorname{Re}(\alpha) \leq 0. \end{aligned} \quad (2)$$

Later on, a number of different modifications of them have been studied by many authors ([3], [11], [18], [19]).

Recently, the first author in a joint paper [1] has considered the following operators involving the Gauss hypergeometric function

$${}_a I_x^{\alpha,\beta,\eta,\mu} f = \frac{x^{-\alpha-\beta-2\mu}}{\Gamma(\alpha)} \int_a^x t^\mu (x-t)^{\alpha-1} {}_2F_1 \left(\alpha + \beta + \mu, -\eta; \alpha; 1 - \frac{t}{x} \right) f(t) dt, \quad (3)$$

where $\operatorname{Re}(\alpha)$, $\operatorname{Re}(\beta) > 0$, and η and μ are real numbers.

In this paper we consider the Erdélyi-Kober operator ${}_a I_x^{\eta, \alpha}$ with a nonzero lower limit of integration $a > 0$ [6,19]

$$\begin{aligned} {}_a I_x^{\eta, \alpha} f(x) &= \frac{x^{-\eta-\alpha}}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} t^\eta f(t) dt, \quad Re(\alpha) > 0, \\ &= x^{-\eta-\alpha} \frac{d^n}{dx^n} x^{\eta+\alpha+n} {}_a I_x^{\eta, \alpha+n} f(x), \quad -n < Re(\alpha) \leq 0, \end{aligned} \quad (4)$$

and the composition of those operators with different lower limit of integration a and b , $0 < a \leq b$,

$${}_b I_x^{\gamma, \beta} {}_a I_x^{\eta, \alpha} f(x)$$

for the cases $Re(\alpha), Re(\beta) > 0$, and $Re(\alpha) \leq 0 < Re(\beta)$.

It is easy to see that the Erdélyi-Kober operator ${}_a I_x^{\eta, \alpha}$ can be expressed through the Riemann-Liouville operator (1)

$${}_a I_x^{\eta, \alpha} f(x) = x^{-\alpha-\eta} {}_a I_x^\alpha x^\eta f(x). \quad (5)$$

Particular case, when $a = b > 0$, are also obtained.

2. ${}_a I_x^{\eta, \alpha}$ for $Re(\alpha) \leq 0$

Let $-n < Re(\alpha) \leq 0$, and function f be n times differentiable on an interval $[a, a+r)$, $r > 0$. We record Taylor's expansion for the function $f(x)$ near the point a

$$f(x) = \sum_{j=0}^{n-1} \frac{f^{(j)}(a)}{j!} (x-a)^j + \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} f^{(n)}(t) dt, \quad a \leq x < a+r. \quad (6)$$

From (4) and (6), for $-n < Re(\alpha) \leq 0$, we have

$$\begin{aligned} {}_a I_x^{\eta, \alpha} f(x) &= \sum_{j=0}^{n-1} \frac{f^{(j)}(a)}{j!} {}_a I_x^{\eta, \alpha} (x-a)^j \\ &+ \frac{x^{-\eta-\alpha}}{(n-1)! \Gamma(\alpha+n)} \frac{d^n}{dx^n} \int_a^x y^\eta (x-y)^{\alpha+n-1} \int_a^y (y-t)^{n-1} f^{(n)}(t) dt dy. \end{aligned} \quad (7)$$

Applying the relation (5) between the Erdélyi-Kober operator (4) and the Riemann-Liouville operator (1), and formula (2.47) from [19]

$${}_a I_x^\alpha x^\eta (x-a)^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} a^\eta (x-a)^{\alpha+\gamma} {}_2F_1 \left(-\eta, \gamma+1; \alpha+\gamma+1; 1 - \frac{x}{a} \right), \quad (8)$$

we obtain

$$\begin{aligned} {}_a I_x^{\eta, \alpha} f(x) &= a^\eta x^{-\eta-\alpha} \sum_{j=0}^{n-1} \frac{f^{(j)}(a)}{\Gamma(\alpha+j+1)} (x-a)^{\alpha+j} {}_2F_1 \left(-\eta, j+1; \alpha+j+1; 1-\frac{x}{a} \right) \\ &\quad + \frac{x^{-\eta-\alpha}}{(n-1)! \Gamma(\alpha+n)} \frac{d^n}{dx^n} \int_a^x (x-y)^{\alpha+n-1} y^\eta \int_a^y (y-t)^{n-1} f^{(n)}(t) dt dy. \end{aligned} \quad (9)$$

Interchanging the order of integration in formula (9) we get

$$\begin{aligned} &\frac{x^{-\eta-\alpha}}{(n-1)! \Gamma(\alpha+n)} \frac{d^n}{dx^n} \int_a^x (x-y)^{\alpha+n-1} y^\eta \int_a^y (y-t)^{n-1} f^{(n)}(t) dt dy \\ &= \frac{x^{-\eta-\alpha}}{(n-1)! \Gamma(\alpha+n)} \frac{d^n}{dx^n} \int_a^x f^{(n)}(t) \int_t^x (x-y)^{\alpha+n-1} (y-t)^{n-1} y^\eta dy dt. \end{aligned} \quad (10)$$

Formula (8) yields

$$\begin{aligned} &\int_t^x (x-y)^{\alpha+n-1} (y-t)^{n-1} y^\eta dy = \Gamma(\alpha+n) {}_t I_x^{\alpha+n} x^\eta (x-t)^{n-1} \\ &= \frac{(n-1)! \Gamma(\alpha+n)}{\Gamma(\alpha+2n)} (x-t)^{\alpha+2n-1} t^\eta {}_2F_1 \left(-\eta, n; \alpha+2n; 1-\frac{x}{t} \right). \end{aligned} \quad (11)$$

Hence,

$$\begin{aligned} &\frac{x^{-\eta-\alpha}}{(n-1)! \Gamma(\alpha+n)} \frac{d^n}{dx^n} \int_a^x (x-y)^{\alpha+n-1} y^\eta \int_a^y (y-t)^{n-1} f^{(n)}(t) dt dy \\ &= \frac{x^{-\eta-\alpha}}{\Gamma(\alpha+2n)} \frac{d^n}{dx^n} \int_a^x (x-t)^{\alpha+2n-1} t^\eta {}_2F_1 \left(-\eta, n; \alpha+2n; 1-\frac{x}{t} \right) f^{(n)}(t) dt. \end{aligned} \quad (12)$$

Since

$$\frac{d^k}{dx^k} \left[(x-t)^{\alpha+2n-1} t^\eta {}_2F_1 \left(-\eta, n; \alpha+2n; 1-\frac{x}{t} \right) \right] \Big|_{t=x} = 0$$

for $k = 0, 1, \dots, n-1$, we have

$$\begin{aligned} &\frac{d^n}{dx^n} \int_a^x (x-t)^{\alpha+2n-1} t^\eta {}_2F_1 \left(-\eta, n; \alpha+2n; 1-\frac{x}{t} \right) f^{(n)}(t) dt \\ &= \int_a^x \frac{d^n}{dx^n} \left[(x-t)^{\alpha+2n-1} t^\eta {}_2F_1 \left(-\eta, n; \alpha+2n; 1-\frac{x}{t} \right) \right] f^{(n)}(t) dt. \end{aligned}$$

Consequently, applying formula (8) again we get

$$\begin{aligned}
& \frac{x^{-\eta-\alpha}}{(n-1)!\Gamma(\alpha+n)} \frac{d^n}{dx^n} \int_a^x f^{(n)}(t) \int_t^x (x-y)^{\alpha+n-1} (y-t)^{n-1} y^n dy dt \\
= & \frac{x^{-\eta-\alpha}}{(n-1)!\Gamma(\alpha+n)} \int_a^x f^{(n)}(t) \frac{d^n}{dx^n} \int_t^x (x-y)^{\alpha+n-1} (y-t)^{n-1} y^n dy dt \\
= & \frac{x^{-\eta-\alpha}}{(n-1)!} \int_a^x f^{(n)}(t) \frac{d^n}{dx^n} {}_tI_x^{\alpha+n} ((x-t)^{n-1} x^n) dt \\
= & \frac{x^{-\eta-\alpha}}{(n-1)!} \int_a^x f^{(n)}(t) {}_tI_x^\alpha ((x-t)^{n-1} x^n) dt \\
= & \frac{x^{-\eta-\alpha}}{\Gamma(\alpha+n)} \int_a^x (x-t)^{\alpha+n-1} t^\eta {}_2F_1 \left(-\eta, n; \alpha+n; 1-\frac{x}{t} \right) f^{(n)}(t) dt. \quad (13)
\end{aligned}$$

Formulas (9) and (13) yield

$$\begin{aligned}
{}_aI_x^{\eta,\alpha} f(x) &= a^\eta x^{-\eta-\alpha} \sum_{j=0}^{n-1} \frac{f^{(j)}(a)}{\Gamma(\alpha+j+1)} (x-a)^{\alpha+j} {}_2F_1 \left(-\eta, j+1; \alpha+j+1; 1-\frac{x}{a} \right) \\
&+ \frac{x^{-\eta-\alpha}}{\Gamma(\alpha+n)} \int_a^x (x-t)^{\alpha+n-1} t^\eta {}_2F_1 \left(-\eta, n; \alpha+n; 1-\frac{x}{t} \right) f^{(n)}(t) dt. \quad (14)
\end{aligned}$$

The transformation [15]

$${}_2F_1(a, b; c; z) = (1-z)^{-a} {}_2F_1 \left(a, c-b; c; \frac{z}{z-1} \right) \quad (15)$$

for the Gauss hypergeometric function leads to

$$\begin{aligned}
{}_aI_x^{\eta,\alpha} f(x) &= a^\eta x^{-\eta-\alpha} \sum_{j=0}^{n-1} f^{(j)}(a) \frac{(x-a)^{\alpha+j}}{\Gamma(\alpha+j+1)} {}_2F_1 \left(-\eta, j+1; \alpha+j+1; 1-\frac{x}{a} \right) \\
&+ {}_aI_x^{\alpha+n, -n, \eta} f^{(n)}(x); \quad Re(n+\alpha) > 0, \quad (16)
\end{aligned}$$

where

$${}_aI_x^{\alpha,\beta,\eta} f = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} {}_2F_1 \left(\alpha+\beta, -\eta; \alpha; 1-\frac{t}{x} \right) f(t) dt \quad (17)$$

is the Saigo operator ([18],[19]) with the lower limit of integration equal to a instead of zero.

3. ${}_b I_x^{\gamma, \beta} {}_a I_x^{\eta, \alpha}$ for $Re(\alpha) > 0$ and $Re(\beta) > 0$

Peña [14] has established the following formula for the composition of Riemann-Liouville operators with different lower limits of integration

$${}_b I_x^{\beta} {}_a I_x^{\alpha} f(x) = \frac{(x-b)^{\beta}}{\Gamma(\alpha)\Gamma(\beta+1)} \sum_{k=0}^{\infty} \frac{(1-\alpha)_k (\beta)_k}{(\beta+1)_k k!} (x-b)^k \int_a^b (x-y)^{\alpha-k-1} f(y) dy + {}_b I_x^{\alpha+\beta} f(x), \quad Re(\alpha) > 0. \quad (18)$$

Nahushev and Salahitdinov [12] also considered the law of composition for Chen fractional integral operators [19] with different origins.

First, we consider the composition of operators ${}_b I_x^{\gamma, \beta}$ and ${}_a I_x^{\eta, \alpha}$ for $Re(\alpha), Re(\beta) > 0$, and $0 < a \leq b$.

We have

$${}_b I_x^{\gamma, \beta} {}_a I_x^{\eta, \alpha} f(x) = \frac{x^{-\gamma-\beta}}{\Gamma(\alpha)\Gamma(\beta)} \int_b^x (x-t)^{\beta-1} t^{\gamma-\eta-\alpha} \int_a^t (t-y)^{\alpha-1} y^{\eta} f(y) dy dt.$$

Interchanging the order of integration we obtain

$${}_b I_x^{\gamma, \beta} {}_a I_x^{\eta, \alpha} f(x) = \frac{x^{-\gamma-\beta}}{\Gamma(\alpha)\Gamma(\beta)} \left\{ \int_a^b y^{\eta} f(y) \int_b^x t^{\gamma-\eta-\alpha} (x-t)^{\beta-1} (t-y)^{\alpha-1} dt dy + \int_b^x y^{\eta} f(y) \int_y^x t^{\gamma-\eta-\alpha} (x-t)^{\beta-1} (t-y)^{\alpha-1} dt dy \right\}. \quad (19)$$

Making the change of variable $w = \frac{x-t}{x-b}$ and using the integral representation for the Appell function F_1 [2, p. 287 (3.211)]

$$\begin{aligned} F_1(a, b, b'; c; x, y) &= \sum_{m, n=0}^{\infty} \frac{(a)_{m+n} (b)_m (b')_n}{(c)_{m+n} m! n!} x^m y^n, \quad |x|, |y| < 1, \\ &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 w^{a-1} (1-w)^{c-a-1} (1-wx)^{-b} (1-wy)^{-b'} dw, \\ & \quad Re(c) > Re(a) > 0, \end{aligned} \quad (20)$$

we obtain

$$\int_b^x t^{\gamma-\eta-\alpha} (x-t)^{\beta-1} (t-y)^{\alpha-1} dt = x^{\gamma-\eta-\alpha} (x-b)^{\beta} (x-y)^{\alpha-1}$$

$$\begin{aligned}
& \int_0^1 w^{\beta-1} \left[1 - \left(\frac{x-b}{x} \right) w \right]^{\gamma-\eta-\alpha} \left[1 - \left(\frac{x-b}{x-y} \right) w \right]^{\alpha-1} dw \\
&= \frac{1}{\beta} x^{\gamma-\eta-\alpha} (x-b)^\beta (x-y)^{\alpha-1} F_1 \left(\beta, \alpha + \eta - \gamma, 1 - \alpha; \beta + 1; \frac{x-b}{x}, \frac{x-b}{x-y} \right).
\end{aligned} \tag{21}$$

On the other hand, formula (8) and transformation (15) yield

$$\begin{aligned}
& \int_y^x t^{\gamma-\eta-\alpha} (x-t)^{\beta-1} (t-y)^{\alpha-1} dt \\
&= \Gamma(\beta) {}_y I_x^\beta (x^{\gamma-\eta-\alpha} (x-y)^{\alpha-1}) \\
&= \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} y^{\gamma-\eta-\alpha} (x-y)^{\alpha+\beta-1} {}_2F_1 \left(\alpha + \eta - \gamma, \alpha; \alpha + \beta; 1 - \frac{x}{y} \right) \\
&= \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} x^{\gamma-\eta-\alpha} (x-y)^{\alpha+\beta-1} {}_2F_1 \left(\alpha + \eta - \gamma, \beta; \alpha + \beta; 1 - \frac{y}{x} \right).
\end{aligned} \tag{22}$$

Therefore

$$\begin{aligned}
& {}_b I_x^{\gamma,\beta} {}_a I_x^{\eta,\alpha} f(x) = \frac{x^{-\alpha-\beta-\eta} (x-b)^\beta}{\Gamma(\alpha)\Gamma(\beta+1)} \\
& \int_a^b (x-y)^{\alpha-1} y^\eta F_1 \left(\beta, \eta + \alpha - \gamma, 1 - \alpha; \beta + 1; 1 - \frac{b}{x}, \frac{x-b}{x-y} \right) f(y) dy \\
& + \frac{x^{-\alpha-\beta-\eta}}{\Gamma(\alpha+\beta)} \int_b^x (x-y)^{\alpha+\beta-1} y^\eta {}_2F_1 \left(\alpha + \eta - \gamma, \beta; \alpha + \beta; 1 - \frac{y}{x} \right) f(y) dy.
\end{aligned} \tag{23}$$

Consequently, using notation (3) we obtain

$$\begin{aligned}
& {}_b I_x^{\gamma,\beta} {}_a I_x^{\eta,\alpha} f(x) = \frac{x^{-\alpha-\beta-\eta} (x-b)^\beta}{\Gamma(\alpha)\Gamma(\beta+1)} \\
& \int_a^b (x-y)^{\alpha-1} y^\eta F_1 \left(\beta, \eta + \alpha - \gamma, 1 - \alpha; \beta + 1; 1 - \frac{b}{x}, \frac{x-b}{x-y} \right) f(y) dy \\
& + x^{\eta-\gamma-\beta} {}_b I_x^{\alpha+\beta, -\beta-\gamma, -\beta, \eta} f(x).
\end{aligned} \tag{24}$$

In particular, if $a = b > 0$, we have

$${}_a I_x^{\gamma,\beta} {}_a I_x^{\eta,\alpha} f(x) = x^{\eta-\gamma-\beta} {}_a I_x^{\alpha+\beta, -\beta-\gamma, -\beta, \eta} f(x).$$

4. ${}_bI_x^{\gamma,\beta} {}_aI_x^{\eta,\alpha}$ for $Re(\alpha) \leq 0 < Re(\beta)$

We consider now the composition of operators ${}_bI_x^{\gamma,\beta}$ and ${}_aI_x^{\eta,\alpha}$ for $Re(\alpha) \leq 0 < Re(\beta)$. Let $n > -Re(\alpha)$ and f be n -times differentiable.

From (16) we have

$$\begin{aligned} {}_bI_x^{\gamma,\beta} {}_aI_x^{\eta,\alpha} f(x) &= \frac{a^\eta x^{-\gamma-\beta}}{\Gamma(\beta)} \sum_{j=0}^{n-1} \frac{f^{(j)}(a)}{\Gamma(\alpha+j+1)} \\ &\int_b^x t^{\gamma-\eta-\alpha} (x-t)^{\beta-1} (t-a)^{\alpha+j} {}_2F_1 \left(-\eta, j+1; \alpha+j+1; 1-\frac{t}{a} \right) dt \\ &+ {}_bI_x^{\gamma,\beta} {}_aI_x^{\alpha+n,-n,\eta} f^{(n)}(x). \end{aligned} \quad (25)$$

Using the transformation formula (15), expressing the hypergeometric function in the series form, and making a simple change of variables ($w = \frac{x-t}{x-b}$), we get

$$\begin{aligned} &\int_b^x t^{\gamma-\eta-\alpha} (x-t)^{\beta-1} (t-a)^{\alpha+j} {}_2F_1 \left(-\eta, j+1; \alpha+j+1; 1-\frac{t}{a} \right) dt \\ &= a^{-\eta} \int_b^x t^{\gamma-\alpha} (x-t)^{\beta-1} (t-a)^{\alpha+j} {}_2F_1 \left(-\eta, \alpha; \alpha+j+1; 1-\frac{a}{t} \right) dt \\ &= a^{-\eta} \sum_{k=0}^{\infty} \frac{(-\eta)_k (\alpha)_k}{(\alpha+j+1)_k k!} (x-b)^\beta x^{\gamma-\alpha-k} (x-a)^{\alpha+j+k} \\ &\int_0^1 w^{\beta-1} \left[1 - \left(\frac{x-b}{x} \right) w \right]^{\gamma-\alpha-k} \left[1 - \left(\frac{x-b}{x-a} \right) w \right]^{\alpha+j+k} dw \\ &= \frac{1}{\beta} a^{-\eta} \sum_{k=0}^{\infty} \frac{(-\eta)_k (\alpha)_k}{(\alpha+j+1)_k k!} x^{\gamma-\alpha-k} (x-b)^\beta (x-a)^{\alpha+j+k} \\ &F_1 \left(\beta, \alpha+k-\gamma, -\alpha-j-k; \beta+1; \frac{x-b}{x}, \frac{x-b}{x-a} \right). \end{aligned} \quad (26)$$

Formulas (3) and (4) yield

$$\begin{aligned} {}_bI_x^{\gamma,\beta} {}_aI_x^{\alpha+n,-n,\eta} f^{(n)}(x) &= \frac{x^{-\gamma-\beta}}{\Gamma(\alpha+n)\Gamma(\beta)} \int_b^x (x-t)^{\beta-1} t^{\gamma-\alpha} \\ &\int_a^t (t-y)^{\alpha+n-1} {}_2F_1 \left(\alpha, -\eta; \alpha+n; 1-\frac{y}{t} \right) f^{(n)}(y) dy dt = \frac{x^{-\gamma-\beta}}{\Gamma(\alpha+n)\Gamma(\beta)} \end{aligned} \quad (27)$$

$$\left[\int_b^x f^{(n)}(y) \int_y^x (x-t)^{\beta-1} t^{\gamma-\alpha} (t-y)^{\alpha+n-1} {}_2F_1\left(\alpha, -\eta; \alpha+n; 1-\frac{y}{t}\right) dt dy \right. \\ \left. + \int_a^b f^{(n)}(y) \int_b^x (x-t)^{\beta-1} t^{\gamma-\alpha} (t-y)^{\alpha+n-1} {}_2F_1\left(\alpha, -\eta; \alpha+n; 1-\frac{y}{t}\right) dt dy \right].$$

We have

$$\begin{aligned} & \int_y^x (x-t)^{\beta-1} t^{\gamma-\alpha} (t-y)^{\alpha+n-1} {}_2F_1\left(\alpha, -\eta; \alpha+n; 1-\frac{y}{t}\right) dt \\ &= \int_y^x (x-t)^{\beta-1} t^{\gamma-\alpha} (t-y)^{\alpha+n-1} \sum_{k=0}^{\infty} \frac{(-\eta)_k (\alpha)_k}{(\alpha+n)_k k!} \left(1-\frac{y}{t}\right)^k dt \\ &= \sum_{k=0}^{\infty} \frac{(-\eta)_k (\alpha)_k}{(\alpha+n)_k k!} \int_y^x (x-t)^{\beta-1} t^{\gamma-\alpha-k} (t-y)^{\alpha+k+n-1} dt \\ &= \Gamma(\beta) \sum_{k=0}^{\infty} \frac{(-\eta)_k (\alpha)_k}{(\alpha+n)_k k!} y I_x^\beta (x^{\gamma-\alpha-k} (x-y)^{\alpha+k+n-1}) \\ &= \Gamma(\beta) \sum_{k=0}^{\infty} \frac{(-\eta)_k (\alpha)_k}{(\alpha+n)_k k!} \frac{\Gamma(\alpha+k+n)}{\Gamma(\alpha+\beta+k+n)} y^{\gamma-\alpha-k} (x-y)^{\alpha+\beta+k+n-1} \\ & \quad {}_2F_1\left(k+\alpha-\gamma, \alpha+k+n; \alpha+\beta+k+n; 1-\frac{x}{y}\right) \\ &= \frac{\Gamma(\beta)\Gamma(\alpha+n)}{\Gamma(\alpha+\beta+n)} \sum_{k=0}^{\infty} \frac{(-\eta)_k (\alpha)_k}{(\alpha+\beta+n)_k k!} x^{\gamma-\alpha-k} (x-y)^{\alpha+\beta+k+n-1} \\ & \quad {}_2F_1\left(k+\alpha-\gamma, \beta; \alpha+\beta+k+n; 1-\frac{y}{x}\right). \quad (28) \end{aligned}$$

On the other hand,

$$\begin{aligned} & \int_b^x (x-t)^{\beta-1} t^{\gamma-\alpha} (t-y)^{\alpha+n-1} {}_2F_1\left(\alpha, -\eta; \alpha+n; 1-\frac{y}{t}\right) dt \\ &= \int_b^x (x-t)^{\beta-1} t^{\gamma-\alpha} (t-y)^{\alpha+n-1} \sum_{k=0}^{\infty} \frac{(-\eta)_k (\alpha)_k}{(\alpha+n)_k k!} \left(1-\frac{y}{t}\right)^k dt \\ &= \sum_{k=0}^{\infty} \frac{(-\eta)_k (\alpha)_k}{(\alpha+n)_k k!} \int_b^x (x-t)^{\beta-1} t^{\gamma-\alpha-k} (t-y)^{\alpha+k+n-1} dt \\ &= \frac{1}{\beta} \sum_{k=0}^{\infty} \frac{(-\eta)_k (\alpha)_k}{(\alpha+n)_k k!} x^{\gamma-\alpha-k} (x-b)^\beta (x-y)^{\alpha+k+n-1} \end{aligned}$$

$$F_1 \left(\beta, \alpha + k - \gamma, 1 - \alpha - n - k; \beta + 1; 1 - \frac{b}{x}, \frac{x - b}{x - y} \right). \quad (29)$$

Consequently,

$$\begin{aligned} {}_b I_x^{\gamma, \beta} {}_a I_x^{\eta, \alpha} f(x) &= \frac{x^{-\alpha - \beta} (x - a)^\alpha (x - b)^\beta}{\Gamma(\beta + 1)} \sum_{j=0}^{n-1} f^{(j)}(a) (x - a)^j \\ &\sum_{k=0}^{\infty} \frac{(-\eta)_k (\alpha)_k}{\Gamma(\alpha + j + k + 1) k!} \left(1 - \frac{a}{x}\right)^k \\ &F_1 \left(\beta, \alpha + k - \gamma, -\alpha - j - k; \beta + 1; 1 - \frac{b}{x}, \frac{x - b}{x - a} \right) \\ &+ x^{-\beta - \gamma} \sum_{k=0}^{\infty} \frac{(-\eta)_k (\alpha)_k}{k!} {}_b I_x^{\alpha + \beta + n + k, -\beta - \gamma - n, -\beta} f^{(n)}(x) \\ &+ \frac{x^{-\alpha - \beta} (x - b)^\beta}{\Gamma(\alpha + n) \Gamma(\beta + 1)} \sum_{k=0}^{\infty} \frac{(-\eta)_k (\alpha)_k}{(\alpha + n)_k k!} x^{-k} \int_a^b (x - y)^{\alpha + k + n - 1} \\ &F_1 \left(\beta, \alpha + k - \gamma, 1 - \alpha - n - k; \beta + 1; 1 - \frac{b}{x}, \frac{x - b}{x - y} \right) f^{(n)}(y) dy. \end{aligned} \quad (30)$$

Using the formula [2]

$$F_1(a, b, b'; c, x, 1) = \frac{\Gamma(c) \Gamma(c - a - b')}{\Gamma(c - a) \Gamma(c - b')} {}_2F_1(a, b; c - b'; x),$$

the representation (30) in case $a = b > 0$ becomes

$$\begin{aligned} {}_a I_x^{\gamma, \beta} {}_a I_x^{\eta, \alpha} f(x) &= x^{-\alpha - \beta} \sum_{j=0}^{n-1} f^{(j)}(a) \frac{(x - a)^{\alpha + \beta + j}}{\Gamma(\alpha + \beta + j + 1)} \\ &\sum_{k=0}^{\infty} \frac{(-\eta)_k (\alpha)_k}{(\alpha + \beta + j + 1)_k k!} \left(1 - \frac{a}{x}\right)^k \\ &{}_2F_1 \left(\beta, \alpha + k - \gamma; \alpha + \beta + j + k + 1; 1 - \frac{a}{x} \right) \\ &+ x^{-\beta - \gamma} \sum_{k=0}^{\infty} \frac{(-\eta)_k (\alpha)_k}{k!} {}_a I_x^{\alpha + \beta + n + k, -\beta - \gamma - n, -\beta} f^{(n)}(x). \end{aligned} \quad (31)$$

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