GENERALIZED ELLIPTIC-TYPE INTEGRALS AND ASYMPTOTIC FORMULAS

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Abstract

A number of families of elliptic-type integrals have been studied recently due to their importance and potential for applications in some problems of radiation physics. The object of this work is to present a unified and generalized form of such elliptic-type integrals and to study its properties, including recurrence formulas and asymptotic expansion.

Keywords: Elliptic-type integrals, asymptotic formulas, recurrence relations.

1. Introduction

Elliptic-type integrals occur in a number of physical problems and often in the form of multiple integrals [1, 2]. For example, the problems dealing with the computation of the radiation field off axis from certain uniform circular disc radiating according to an arbitrary angular distribution law [3], when treated with Legendre polynomials expansion method, give rise to the Epstein and Hubbell [4, 5] family of elliptic-type integrals:

$$\Omega_j(k) = \int_0^{\pi} \left(1 - k^2 \cos\theta\right)^{-j-\frac{1}{2}} d\theta \quad ; \quad j = 0, 1, 2, \dots , \tag{1.1}$$

and $0 \le k < 1$.

Elliptic integrals (1.1) have been studied and generalized by many authors. Here we select and give a brief review of the most relevant results. A detailed survey of elliptic-type integrals can be found in a recent paper of Al-Zamel and Kalla [6].

Kalla, Conde, and Hubbell [7] have studied a family of integrals of the form

$$R_\mu\left(k, \alpha, \gamma\right) = \int_0^{\pi} \cos^{2\alpha-1}\left(\frac{\theta}{2}\right) \sin^{2\mu-2\alpha-1}\left(\frac{\theta}{2}\right) \left(1 - k^2 \cos\theta\right)^{-\mu-\frac{1}{2}} d\theta, \tag{1.2}$$

where $0 \le k < 1$, $\quad Re(\gamma) > Re(\alpha) \ge 0, \quad Re(\mu) > -\frac{1}{2}$.

The integrals (1.2) generalize the elliptic integrals (1.1) in the sense that

$$R_j\left(k, \frac{1}{2}, 1\right) = \Omega_j(k), \quad j = 0, 1, 2, \dots, \quad 0 \le k < 1.$$

Srivastava and Siddiqi [8] gave a unified presentation of certain families of elliptic-type integrals in the form:

$$\Lambda_{(\lambda,\mu)}^{(\alpha,\beta)}(\rho; k) = \int_0^\pi \cos^{2\alpha-1}\left(\frac{\theta}{2}\right) \sin^{2\beta-1}\left(\frac{\theta}{2}\right) \left(1 - k^2\cos\theta\right)^{-\mu-\frac{1}{2}} \left(1 - \rho\sin^2\frac{\theta}{2}\right)^{-\lambda} d\theta, \tag{1.3}$$

where,

$$0 \le k < 1, \quad Re(\alpha), \quad Re(\beta) > 0, \ \lambda, \mu \in \mathbb{C}, \quad |\rho| < 1.$$

It can easily be shown that integrals (1.2) and hence (1.1) are special cases of (1.3) by verifying that

$$\Lambda_{(0,\mu)}^{(\alpha,\gamma-\alpha)}(\rho; k) = R_\mu(k, \alpha, \gamma),$$

$$\Lambda_{(0,j)}^{\left(\frac{1}{2},\frac{1}{2}\right)}(\rho; k) = \Omega_j(k) \quad ; \quad j = 0, 1, 2, \dots,$$

and $0 \le k < 1$.

In 1996, Kalla and Tuan [9] have defined yet another generalization of elliptic-type integrals of the form

$$\Lambda_{(\lambda,\gamma,\mu)}^{(\alpha,\beta)}(\rho, \delta; k) = \int_0^\pi \cos^{2\alpha-1}\left(\frac{\theta}{2}\right) \sin^{2\beta-1}\left(\frac{\theta}{2}\right) \left(1 - k^2\cos\theta\right)^{-\mu-\frac{1}{2}}$$

$$\left(1 - \rho\sin^2\frac{\theta}{2}\right)^{-\lambda} \left(1 + \delta\cos^2\frac{\theta}{2}\right)^{-\gamma} d\theta, \tag{1.4}$$

where $0 \leq k < 1,\ \ Re(\alpha),\ \ Re(\beta) > 0,\ \ \lambda, \mu, \gamma \in \mathbb{C}$, and either $|\rho|,\ |\delta| < 1$ or ρ (or δ) $\in \mathbb{C}$ whenever $\lambda = m$ or $\gamma = -m,\ \ m \in N_0$, respectively.

Again, elliptic-type integrals defined by equation (1.3) and hence (1.2) and (1.1) are special cases of (1.4) as we can see that

$$\Lambda^{(\alpha,\beta)}_{(\lambda,\mu,\gamma)}(\rho, 0; k) = \Lambda^{(\alpha,\beta)}_{(\lambda,\mu)}(\rho; k), \tag{1.5}$$

$$\Lambda^{(\alpha,\gamma-\alpha)}_{(0,\gamma,\mu)}(\rho, 0; k) = \Lambda^{(\alpha,\gamma-\alpha)}_{(0,\mu)}(\rho; k) = R_\mu\left(k, \alpha, \gamma\right), \tag{1.6}$$

$$\Lambda^{(\frac{1}{2},\frac{1}{2})}_{(0,0,j)}(0, 0; k) = R_j\left(k, \frac{1}{2}, 1\right) = \Omega_j(k), \quad j \geq 0. \tag{1.7}$$

In this paper, we present a unified and generalized form of elliptic-type integrals potentially useful in radiation field problems. The generalized family of elliptic-type integrals is expresed in terms of the Lauricella hypergeometric function of n variables $F^{(n)}_D$ [10]. A number of recurrence relations are derived and some special cases are mentioned. We obtain its asymptotic expansion as $k^2 \longrightarrow 1$.

2. Definition and Explicit Representation

Here we consider a unified and generalized form of a family of elliptic-type integrals:

$$\begin{aligned} Z_{(\gamma)}^{(\alpha,\beta)}(k) &= Z_{(\gamma_j)}^{(\alpha,\beta)}(k_j) = Z_{(\gamma_1,\gamma_2,\cdots,\gamma_n)}^{(\alpha,\beta)}(k_1,k_2,\cdots,k_n) \\ &= \int_0^{\pi} \cos^{2\alpha-1}\left(\frac{\theta}{2}\right)\sin^{2\beta-1}\left(\frac{\theta}{2}\right)\prod_{j=1}^{n}\left(1-k_j^2\cos\theta\right)^{-\gamma_j} d\theta, \quad (2.1) \end{aligned}$$

where

$$Re(\alpha),\ Re(\beta) > 0, \quad |k_j| < 1, \quad \gamma_j \in \mathbb{C}, \quad j = 1, 2, \ldots, n,$$

Special Cases:

Case 1: Let $n = 1$, $k_1 = k$, $\gamma_1 = j + \frac{1}{2}$, and $\alpha = \beta = \frac{1}{2}$ in (2.1). We get

$$Z_{(j+\frac{1}{2})}^{(\frac{1}{2},\frac{1}{2})}(k) = \int_0^{\pi}\left(1-k^2\cos\theta\right)^{-j-\frac{1}{2}} d\theta = \Omega_j(k), \quad j = 0, 1, 2, \ldots, \qquad (2.2)$$

where $\Omega_j(k)$ is the Epstein and Hubbell family of elliptic type integrals defined by equation (1.1).

Case 2: Let $n = 1$, $k_1 = k$, $\gamma_1 = \mu + \frac{1}{2}$, and $\beta = \gamma - \alpha$ in (2.1). We get

$$\begin{aligned} Z_{(\mu+\frac{1}{2})}^{(\alpha,\gamma-\alpha)}(k) &= \int_0^{\pi}\cos^{2\alpha-1}\left(\frac{\theta}{2}\right)\sin^{2\gamma-2\alpha-1}\left(\frac{\theta}{2}\right)\left(1-k^2\cos\theta\right)^{-\mu-\frac{1}{2}} d\theta \\ &= R_{\mu}(k,\alpha,\gamma). \qquad (2.3) \end{aligned}$$

Case 3: Let $n = 3$, so definition (2.1) will be of the form

$$Z_{(\gamma_1,\gamma_2,\gamma_3)}^{(\alpha,\beta)}(k_1,k_2,k_3) = \int_0^{\pi}\cos^{2\alpha-1}\left(\frac{\theta}{2}\right)\sin^{2\beta-1}\left(\frac{\theta}{2}\right)\left(1-k_1^2\cos\theta\right)^{-\gamma_1}$$

$$\left(1-k_2^2\cos\theta\right)^{-\gamma_2}\left(1-k_3^2\cos\theta\right)^{-\gamma_3}d\theta. \tag{2.4}$$

Now, let

$$\begin{aligned}
\gamma_1 &= \mu+\frac{1}{2},\ \gamma_2=\gamma_3=\lambda,\ k=k_1,\\
\rho &= \frac{2k_2^2}{k_2^2-1}\Rightarrow k_2^2=\frac{\rho}{\rho-2},\\
\delta &= \frac{-2k_3^2}{k_3^2+1}\Rightarrow k_3^2=\frac{-\delta}{2+\delta},
\end{aligned}$$

then from (2.4) we obtain

$$Z^{(\alpha,\beta)}_{(\mu+\frac{1}{2},\lambda,\gamma)}\left(k,\sqrt{\frac{\rho}{\rho-2}},\sqrt{\frac{-\delta}{\delta+2}}\right)=\left(\frac{2}{2-\rho}\right)^{-\lambda}\left(\frac{2}{2+\delta}\right)^{-\gamma}$$

$$\int_0^\pi \cos^{2\alpha-1}\left(\frac{\theta}{2}\right)\sin^{2\beta-1}\left(\frac{\theta}{2}\right)\left(1-k^2\cos\theta\right)^{-\mu-\frac{1}{2}}\left(1-\rho\sin^2\frac{\theta}{2}\right)^{-\lambda}\left(1+\delta\cos^2\frac{\theta}{2}\right)^{-\gamma}d\theta$$

$$=\left(1-\frac{\rho}{2}\right)^{\lambda}\left(1+\frac{\delta}{2}\right)^{\gamma}\Lambda^{(\alpha,\beta)}_{(\mu,\lambda,\nu)}\left(\rho,\delta;k\right). \tag{2.5}$$

We have

$$Z^{(\alpha,\beta)}_{(\gamma)}(k)=\int_0^\pi \cos^{2\alpha-1}\left(\frac{\theta}{2}\right)\sin^{2\beta-1}\left(\frac{\theta}{2}\right)\prod_{j=1}^{n}\left(1-k_j^2\cos\theta\right)^{-\gamma_j}d\theta$$

$$=\int_0^1(1-u)^{\alpha-1}u^{\beta-1}\prod_{j=1}^{n}\left(1-k_j^2\right)^{-\gamma_j}\left(1-\frac{2k_j^2}{k_j^2-1}u\right)^{-\gamma_j}du$$

$$=\prod_{j=1}^{n}\left(1-k_j^2\right)^{-\gamma_j}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

$$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 u^{\beta-1}(1-u)^{\alpha-1}\prod_{j=1}^{n}\left(1-\frac{2k_j^2u}{k_j^2-1}\right)^{-\gamma_j}du. \tag{2.6}$$

Hence,

$$Z^{(\alpha,\beta)}_{(\gamma)}(k)=\prod_{j=1}^{n}\left(1-k_j^2\right)^{-b_j}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

$$F_D^{(n)}\left(\beta; b_1, b_2, \cdots, b_n; \alpha+\beta; \frac{2k_1^2}{k_1^2-1}, \cdots, \frac{2k_n^2}{k_n^2-1}\right), \tag{2.7}$$

where $F_D^{(n)}$ is the Lauricella hypergeometric function of n variables [10].

By letting $n = 3$, $k_1 = k$, $k_2 = \sqrt{\frac{\rho}{\rho-2}}$, $k_3 = \sqrt{\frac{-\delta}{\delta+2}}$, we get

$$\begin{aligned}
Z^{(\alpha,\beta)}_{(\mu+\frac{1}{2},\lambda,\gamma)}\left(k, \sqrt{\frac{\rho}{\rho-2}}, \sqrt{\frac{-\delta}{\delta+2}}\right) &= \left(1-k^2\right)^{-\mu-\frac{1}{2}}\left(1-\frac{\rho}{\rho-2}\right)^{-\lambda}\left(1+\frac{\delta}{\delta+2}\right)^{-\gamma} \\
&\cdot\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} F_D^{(3)}\left(\beta; \lambda, \gamma, \mu+\frac{1}{2}; \alpha+\beta; \rho, \frac{\delta}{1+\delta}, \frac{2k^2}{k^2-1}\right) \\
&= \left(1-k^2\right)^{-\mu-\frac{1}{2}}\left(\frac{2}{2-\rho}\right)^{-\lambda}\left(\frac{2(\delta+1)}{\delta+2}\right)^{-\gamma}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \\
&\quad F_D^{(3)}\left(\beta; \lambda, \gamma, \mu+\frac{1}{2}; \alpha+\beta; \rho, \frac{\delta}{1+\delta}, \frac{2k^2}{k^2-1}\right) \\
&= \frac{(2-\rho)^{\lambda}(2+\delta)^{\gamma}}{2^{\lambda+\gamma}(\delta+1)^{\gamma}(1-k^2)^{\mu+\frac{1}{2}}}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \\
&\quad F_D^{(3)}\left(\beta; \lambda, \gamma, \mu+\frac{1}{2}; \alpha+\beta; \rho, \frac{\delta}{1+\delta}, \frac{2k^2}{k^2-1}\right),
\end{aligned} \tag{2.8}$$

which agrees with a result of Kalla and Tuan [9].

3. Recurrence Formulas

The following recurrence formulas for $Z_\gamma^{(\alpha,\beta)}(k)$ can be easily derived from the definition (2.1) by suitable rearranging the integrand and using elementary trigonometric identities:

$$Z^{(\alpha,\beta)}_{(\gamma)}(k) = \left(1+k_i^2\right) Z^{(\alpha,\beta)}_{(\gamma_1,\cdots,\gamma_i+1,\cdots\gamma_n)}(k) - 2k_i^2 Z^{(\alpha+1,\beta)}_{(\gamma_1,\cdots,\gamma_i+1,\cdots,\gamma_n)}(k) \tag{3.1}$$

for $\quad i = 1, 2, \cdots, n,$

$$Z^{(\alpha,\beta)}_{(\gamma)}(k) = Z^{(\alpha-1,\beta)}_{(\gamma)}(k) - Z^{(\alpha-1,\beta+1)}_{(\gamma)}(k), \tag{3.2}$$

$$Z_{(\gamma)}^{(\alpha,\beta)}(k) = Z_{(\gamma)}^{(\alpha,\beta-1)}(k) - Z_{(\gamma)}^{(\alpha+1,\beta-1)}(k). \tag{3.3}$$

From (3.2) and (3.3) we get

$$Z_{(\gamma)}^{(\alpha-1,\beta)}(k) - Z_{(\gamma)}^{(\alpha-1,\beta+1)}(k) = Z_{(\gamma)}^{(\alpha,\beta-1)}(k) - Z_{(\gamma)}^{(\alpha+1,\beta-1)}(k). \tag{3.4}$$

Let $n = 3$ in (3.1). We get the following result, say, for $i = 1$,

$$Z_{(\gamma_1,\gamma_2,\gamma_3)}^{(\alpha,\beta)}(k_1,k_2,k_3) = \left(1+k_1^2\right) Z_{(\gamma_1+1,\gamma_2,\gamma_3)}^{(\alpha,\beta)}(k_1,k_2,k_3) - 2k_1^2 Z_{(\gamma_1+1,\gamma_2,\gamma_3)}^{(\alpha+1,\beta)}(k_1,k_2,k_3). \tag{3.5}$$

If we set

$$k_1 = k,\ k_2 = \sqrt{\frac{\rho}{\rho-2}}, k_3 = \sqrt{\frac{-\delta}{\delta+2}}, \gamma_1 = \mu + \frac{1}{2}, \gamma_2 = \lambda,\ \text{and}\ \gamma_3 = \gamma,$$

then (3.5) can be rewritten as:

$$Z_{\left(\mu+\frac{1}{2},\lambda,\gamma\right)}^{(\alpha,\beta)}\left(k,\sqrt{\frac{\rho}{\rho-2}},\sqrt{\frac{-\delta}{\delta+2}}\right) = \left(1+k^2\right) Z_{\left(\mu+\frac{3}{2},\lambda,\gamma\right)}^{(\alpha,\beta)}\left(k,\sqrt{\frac{\rho}{\rho-2}},\sqrt{\frac{-\delta}{\delta+2}}\right)$$
$$-2k^2 Z_{\left(\mu+\frac{3}{2},\lambda,\gamma\right)}^{(\alpha,\beta)}\left(k,\sqrt{\frac{\rho}{\rho-2}},\sqrt{\frac{-\delta}{\delta+2}}\right). \tag{3.6}$$

To obtain a recurrence relation for $R_\mu(k,\alpha,\gamma)$, let $n = 1,\ k_1 = k,\ \gamma_1 = \mu + \frac{1}{2}$, and $\beta = \gamma - \alpha$, then (3.5) leads to

$$Z_{\left(\mu+\frac{1}{2}\right)}^{(\alpha,\gamma-\alpha)}(k) = \left(1+k^2\right) Z_{\left(\mu+\frac{3}{2}\right)}^{(\alpha,\gamma-\alpha)}(k) - 2k^2 Z_{\left(\mu+\frac{3}{2}\right)}^{(\alpha+1,\gamma-\alpha)}(k), \tag{3.7}$$

and by virtue of (2.3) we obtain a known result given by Kalla et al. [7, p. 282, eq. (34)],

$$R_\mu(k,\alpha,\gamma) = \left(1+k^2\right) R_{\mu+1}(k,\alpha,\gamma) - 2k^2 R_{\mu+1}(k,\alpha+1,\ \gamma+1). \tag{3.8}$$

As a special case of (3.2) for $n = 3$, we mention,

$$\Lambda^{(\alpha,\beta)}_{(\mu,\lambda,\gamma)}(\rho,\delta;k) = \Lambda^{(\alpha-1,\beta)}_{(\mu,\lambda,\gamma)}(\rho,\delta;k) - \Lambda^{(\alpha-1,\beta+1)}_{(\mu,\lambda,\gamma)}(\rho,\delta;k) \tag{3.9}$$

for the elliptic-type integrals of Kalla and Tuan [9].

Another recurrence formula for $R_\mu(k,\alpha,\gamma)$ can be deduced from formula (3.9), [7, p.281, eq. (33)],

$$R_\mu(k,\alpha,\gamma) = R_\mu(k,\alpha-1,\gamma-1) - R_\mu(k,\alpha-1,\gamma). \tag{3.10}$$

For $n = 3$, equation (3.3) reduces to a formula for $\Lambda^{(\alpha,\beta)}_{(\mu,\lambda,\gamma)}(\rho,\delta;k)$:

$$\Lambda^{(\alpha,\beta)}_{(\mu,\lambda,\gamma)}(\rho,\delta;k) = \Lambda^{(\alpha,\beta-1)}_{(\mu,\lambda,\gamma)}(\rho,\delta;k) - \Lambda^{(\alpha+1,\beta-1)}_{(\mu,\lambda,\gamma)}(\rho,\delta;k), \tag{3.11}$$

and the following known result [7, p.281, eq. (32)], can be easily derived:

$$R_\mu(k,\alpha,\gamma) = R_\mu(k,\alpha,\gamma-1) - R_\mu(k,\alpha+1,\gamma). \tag{3.12}$$

Finally we mention two special cases of (3.4):

$$\Lambda^{(\alpha-1,\beta)}_{(\mu,\lambda,\gamma)}(\rho,\delta;k) - \Lambda^{(\alpha-1,\beta+1)}_{(\mu,\lambda,\gamma)}(\rho,\delta;k) = \Lambda^{(\alpha,\beta-1)}_{(\mu,\lambda,\gamma)}(\rho,\delta;k) - \Lambda^{(\alpha+1,\beta-1)}_{(\mu,\lambda,\gamma)}(\rho,\delta;k), \tag{3.13}$$

and

$$R_\mu(k,\alpha-1,\gamma-1) - R_\mu(k,\alpha,\gamma-1) = R_\mu(k,\alpha-1,\gamma) - R_\mu(k,\alpha+1,\gamma). \tag{3.14}$$

4. ASYMPTOTIC EXPANSION FOR $Z_{(\gamma)}^{(\alpha,\beta)}(k)$

From formula (2.7) we have

$$Z_{(\gamma)}^{(\alpha,\beta)}(k) = B(\alpha,\beta)\prod_{j=1}^{n}\left(1-k_j^2\right)^{-\gamma_j}$$

$$\cdot \sum_{m_1=0}^{\infty}\cdots\sum_{m_{n-1}=0}^{\infty}\frac{(\beta)_{m_1+\cdots+m_{n-1}}(\gamma_1)_{m_1}\ldots(\gamma_{n-1})_{m_{n-1}}}{(\alpha+\beta)_{m_1+\cdots+m_{n-1}}\,m_1!\ldots m_{n-1}!}$$

$$\cdot\left(\frac{2k_1^2}{k_1^2-1}\right)^{m_1}\cdots\left(\frac{2k_{n-1}^2}{k_{n-1}^2-1}\right)^{m_{n-1}}$$

$${}_2F_1\left(\beta+m_1+\cdots+m_{n-1},\gamma_n;\alpha+\beta+m_1+\cdots+m_{n-1};\frac{2k_n^2}{k_n^2-1}\right). \qquad (4.1)$$

Let $\beta-\gamma_n$ be not integer. The analytic continuation formula [1, eq. 15.3.7] for the Gauss hypergeometric function yields

$${}_2F_1\left(\beta+m_1+\cdots+m_{n-1},\gamma_n;\alpha+\beta+m_1+\cdots+m_{n-1};\frac{2k_n^2}{k_n^2-1}\right)$$

$$=\frac{\Gamma(\alpha+\beta+m_1+\cdots+m_{n-1})\,\Gamma(\gamma_n-\beta-m_1-\cdots-m_{n-1})}{\Gamma(\gamma_n)\,\Gamma(\alpha)}\left(\frac{1-k_n^2}{2k_n^2}\right)^{\beta+m_1+\cdots+m_{n-1}}$$

$${}_2F_1\left(\beta+m_1+\cdots+m_{n-1},1-\alpha;1-\gamma_n+\beta+m_1+\cdots+m_{n-1};\ \frac{k_n^2-1}{2k_n^2}\right)$$

$$+\ \frac{\Gamma(\alpha+\beta+m_1+\cdots+m_{n-1})\,\Gamma(\beta-\gamma_n+m_1+\cdots+m_{n-1})}{\Gamma(\beta+m_1+\cdots+m_{n-1})\,\Gamma(\alpha+\beta-\gamma_n+m_1+\cdots+m_{n-1})}\left(\frac{1-k_n^2}{2k_n^2}\right)^{\gamma_n}$$

$${}_2F_1\left(\gamma_n,1+\gamma_n-\alpha-\beta-m_1-\cdots-m_{n-1};1+\gamma_n-\beta-m_1-\cdots-m_{n-1};\ \frac{k_n^2-1}{2k_n^2}\right)$$

$$=\frac{\Gamma(\alpha+\beta)\,\Gamma(\gamma_n-\beta)}{\Gamma(\gamma_n)\,\Gamma(\alpha)}\left(\frac{1-k_n^2}{2k_n^2}\right)^{\beta}\frac{(\alpha+\beta)_{m_1+\cdots+m_{n-1}}}{(1+\beta-\gamma_n)_{m_1+\cdots+m_{n-1}}}\left(\frac{k_n^2-1}{2k_n^2}\right)^{m_1+\cdots+m_{n-1}}$$

$${}_2F_1\left(\beta+m_1+\cdots+m_{n-1},1-\alpha;1+\beta-\gamma_n+m_1+\cdots+m_{n-1};\ \frac{k_n^2-1}{2k_n^2}\right)$$

$$+\frac{\Gamma\left(\alpha+\beta\right)\Gamma\left(\beta-\gamma_n\right)}{\Gamma\left(\beta\right)\Gamma\left(\alpha+\beta-\gamma_n\right)}\left(\frac{1-k_n^2}{2k_n^2}\right)^{\gamma_n}\frac{\left(\alpha+\beta\right)_{m_1+\cdots+m_{n-1}}\left(\beta-\gamma_n\right)_{m_1+\cdots+m_{n-1}}}{\left(\beta\right)_{m_1+\cdots+m_{n-1}}\left(\alpha+\beta-\gamma_n\right)_{m_1+\cdots+m_{n-1}}}$$

$${}_2F_1\left(\gamma_n,1+\gamma_n-\alpha-\beta-m_1-\cdots-m_{n-1};1+\gamma_n-\beta-m_1-\cdots-m_{n-1};\ \frac{k_n^2-1}{2k_n^2}\right). \tag{4.2}$$

Consequently,

$$Z_{(\gamma)}^{(\alpha,\beta)}(k)=\frac{\Gamma(\beta)\Gamma\left(\gamma_n-\beta\right)}{\Gamma\left(\gamma_n\right)}\left(2k_n^2\right)^{-\beta}\left(1-k_n^2\right)^{\beta-\gamma_n}\prod_{j=1}^{n-1}\left(1-k_j^2\right)^{-\gamma_j}$$

$$\sum_{m_1=0}^{\infty}\cdots\sum_{m_{n-1}=0}^{\infty}\frac{(\beta)_{m_1+\cdots+m_{n-1}}\left(\gamma_1\right)_{m_1}\cdots\left(\gamma_{n-1}\right)_{m_{n-1}}}{m_1!\ldots m_{n-1}!\left(1+\beta-\gamma_n\right)_{m_1+\cdots+m_{n-1}}}\left[\frac{k_1^2\left(k_n^2-1\right)}{\left(k_1^2-1\right)k_n^2}\right]^{m_1}\cdots\left[\frac{k_{n-1}^2\left(k_n^2-1\right)}{\left(k_{n-1}^2-1\right)k_n^2}\right]^{m_{n-1}}$$

$${}_2F_1\left(\beta+m_1+\cdots+m_{n-1},1-\alpha;1+\beta-\gamma_n+m_1+\cdots+m_{n-1};\ \frac{k_n^2-1}{2k_n^2}\right)$$

$$+\frac{\Gamma(\alpha)\Gamma\left(\beta-\gamma_n\right)}{\Gamma\left(\alpha+\beta-\gamma_n\right)}\left(2k_n^2\right)^{-\gamma_n}\prod_{j=1}^{n-1}\left(1-k_j^2\right)^{-\gamma_j}$$

$$\sum_{m_1=0}^{\infty}\cdots\sum_{m_{n-1}=0}^{\infty}\frac{\left(\beta-\gamma_n\right)_{m_1+\cdots+m_{n-1}}\left(\gamma_1\right)_{m_1}\cdots\left(\gamma_{n-1}\right)_{m_{n-1}}}{\left(\alpha+\beta-\gamma_n\right)_{m_1+\cdots+m_{n-1}}m_1!\cdots m_{n-1}!}$$

$$\cdot\left(\frac{2k_1^2}{k_1^2-1}\right)^{m_1}\cdots\left(\frac{2k_{n-1}^2}{k_{n-1}^2-1}\right)^{m_{n-1}}$$

$${}_2F_1\left(\gamma_n,1-\alpha-\beta+\gamma_n-m_1-\cdots-m_{n-1};1-\beta+\gamma_n-m_1-\cdots-m_{n-1};\ \frac{k_n^2-1}{2k_n^2}\right). \tag{4.3}$$

Let $\gamma_n=\beta-\ell,\ \ \ell=0,1,2,\ldots.$ Applying formula [1, eq. 15.3.19] we have

$${}_2F_1\left(\beta+m_1+\cdots+m_{n-1},\beta-\ell;\alpha+\beta+m_1+\cdots+m_{n-1};\ \frac{2k_n^2}{k_n^2-1}\right)$$

$$=\frac{\Gamma\left(\alpha+\beta+m_1+\cdots+m_{n-1}\right)}{\Gamma\left(\beta+m_1+\cdots+m_{n-1}\right)\Gamma\left(\alpha+\ell+m_1+\cdots+m_{n-1}\right)}\left(\frac{1-k_n^2}{2k_n^2}\right)^{\beta+m_1+\cdots+m_{n-1}}$$

$$\cdot\sum_{r=0}^{\infty}\frac{(\beta-\ell)_{r+\ell+m_1+\cdots+m_{n-1}}\left(1-\alpha-\ell-m_1-\cdots-m_{n-1}\right)_{r+\ell+m_1+\cdots+m_{n-1}}}{r!\left(r+\ell+m_1+\cdots+m_{n-1}\right)!}$$

$$\left(\frac{k_n^2-1}{2k_n^2}\right)^r \Big[\ln(2k^2)-\ln(1-k^2)+\psi\left(1+r+\ell+m_1+\cdots+m_{n-1}\right)+$$

$$+\psi(1+r)-\psi\left(\beta+r+m_1+\cdots+m_{n-1}\right)-\psi\left(\alpha-r\right)]$$

$$+\ \left(\frac{1-k_n^2}{2k_n^2}\right)^{\beta-\ell}\frac{\Gamma\left(\alpha+\beta+m_1+\cdots+m_{n-1}\right)}{\Gamma\left(\beta+m_1+\cdots+m_{n-1}\right)}$$

$$\cdot\sum_{r=0}^{\ell+m_1+\cdots+m_{n-1}-1}\frac{\Gamma\left(\ell+m_1+\cdots+m_{n-1}-r\right)\left(\beta-\ell\right)_r}{\Gamma\left(\alpha+m_1+\cdots+m_{n-1}+\ell-r\right)\ r!}\left(\frac{k_n^2-1}{2k_n^2}\right)^r$$

$$=\frac{\Gamma\left(\alpha+\beta\right)}{\Gamma(\beta)\Gamma\left(\alpha+\ell\right)}\left(\frac{1-k_n^2}{2k_n^2}\right)^\beta\frac{\left(\alpha+\beta\right)_{m_1+\cdots+m_{n-1}}}{\left(\beta\right)_{m_1+\cdots+m_{n-1}}\left(\alpha+\ell\right)_{m_1+\cdots+m_{n-1}}}\left(\frac{k_n^2-1}{2k_n^2}\right)^{m_1+\cdots+m_{n-1}}$$

$$\sum_{r=0}^{\infty}\frac{\left(1-\beta\right)_\ell\left(\beta\right)_{r+m_1+\cdots+m_{n-1}}\left(1-\alpha\right)_r\left(\alpha\right)_{\ell+m_1+\cdots+m_{n-1}}}{r!\left(r+\ell+m_1+\cdots+m_{n-1}\right)!}\left(\frac{k_n^2-1}{2k_n^2}\right)^r$$

$$\Big[\ln\left(2k^2\right)-\ln\left(1-k^2\right)+\psi\left(1+r+\ell+m_1+\cdots+m_{n-1}\right)+\psi\left(1+r\right)$$

$$-\psi\left(\beta+r+m_1+\cdots+m_{n-1}\right)-\psi\left(\alpha-r\right)]+\frac{\Gamma\left(\alpha+\beta\right)}{\Gamma(\alpha)\Gamma(\beta)}\left(\frac{1-k_n^2}{2k_n^2}\right)^{\beta-\ell}$$

$$\frac{\left(\alpha+\beta\right)_{m_1+\cdots+m_{n-1}}}{\left(\beta\right)_{m_1+\cdots+m_{n-1}}}\sum_{r=0}^{\ell+m_1+\cdots+m_{n-1}-1}\frac{\left(\ell+m_1+\cdots+m_{n-1}-r-1\right)!\ \left(\beta-\ell\right)_r}{\left(\alpha\right)_{m_1+\cdots+m_{n-1}+\ell-r}\,r!}\left(\frac{k_n^2-1}{2k_n^2}\right)^r. \tag{4.4}$$

Consequently,

$$Z^{(\alpha,\beta)}_{(\gamma_1,\cdots,\gamma_{n-1},\beta-\ell)}(k)=\left(1-\beta\right)_\ell\left(2k_n^2\right)^{-\beta}\left(1-k_n^2\right)^\ell\prod_{j=1}^{n-1}\left(1-k_j^2\right)^{-\gamma_j}$$

$$\sum_{m_1=0}^{\infty}\cdots\sum_{m_{n-1}=0}^{\infty}\frac{\left(\gamma_1\right)_{m_1}\ldots\left(\gamma_{n-1}\right)_{m_{n-1}}}{m_1!\ldots m_{n-1}!}\left[\frac{k_1^2\left(k_n^2-1\right)}{k_n^2\left(k_1^2-1\right)}\right]^{m_1}\cdots\left[\frac{k_{n-1}^2\left(k_n^2-1\right)}{k_n^2\left(k_{n-1}^2-1\right)}\right]^{m_{n-1}}$$

$$\sum_{r=0}^{\infty}\frac{\left(\beta\right)_{r+m_1+\cdots+m_{n-1}}\left(1-\alpha\right)_r}{r!\left(r+\ell+m_1+\cdots+m_{n-1}\right)!}\left(\frac{k_n^2-1}{2k_n^2}\right)^r\Big[\ln\left(2k^2\right)-\ln\left(1-k^2\right)$$

$$+\ \psi\left(1+r+\ell+m_1+\cdots+m_{n-1}\right)+\psi(1+r)-\psi\left(\beta+r+m_1+\cdots+m_{n-1}\right)$$

$$-\ \psi\left(\alpha-r\right)]+\left(2k_n^2\right)^{\ell-\beta}\prod_{j=1}^{n-1}\left(1-k_j^2\right)^{-\gamma_j}\sum_{m_1=0}^{\infty}\cdots\sum_{m_{n-1}=0}^{\infty}\frac{\left(\gamma_1\right)_{m_1}\ldots\left(\gamma_{n-1}\right)_{m_{n-1}}}{m_1!\ldots m_{n-1}!}$$

$$\left(\frac{2k_1^2}{k_1^2-1}\right)^{m_1}\cdots\left(\frac{2k_{n-1}^2}{k_{n-1}^2-1}\right)^{m_{n-1}}\sum_{r=0}^{\ell+m_1+\cdots+m_{n-1}-1}\frac{\left(\beta-\ell\right)_r\left(\ell+m_1+\cdots+m_{n-1}-r-1\right)!}{\left(\alpha\right)_{m_1+\cdots+m_{n-1}+\ell-r}\ \ r!}$$

$$\cdot \left(\frac{k_n^2-1}{2k_n^2}\right)^r = (1-\beta)_\ell \left(2k_n^2\right)^{-\beta} \left(1-k_n^2\right)^\ell \prod_{j=1}^{n-1} \left(1-k_j^2\right)^{-\gamma_j}$$

$$\sum_{m_1=0}^{\infty} \cdots \sum_{m_{n-1}=0}^{\infty} \sum_{r=0}^{\infty} \frac{(\beta)_{r+m_1+\cdots+m_{n-1}} (1-\alpha)_r (\gamma_1)_{m_1} \cdots (\gamma_{n-1})_{m_{n-1}}}{(r+\ell+m_1+\cdots+m_{n-1})! \; r! \; m_1! \ldots m_{n-1}!}$$

$$\left[\frac{k_1^2 (k_n^2-1)}{k_n^2 (k_1^2-1)}\right]^{m_1} \cdots \left[\frac{k_{n-1}^2 (k_n^2-1)}{k_n^2 (k_{n-1}^2-1)}\right]^{m_{n-1}} \left[\frac{k_n^2-1}{2k_n^2}\right]^r \left[\ln\left(2k^2\right) - \ln\left(1-k^2\right) + \psi(1+r)\right.$$

$$\left. +\psi\left(1+r+\ell+m_1+\cdots+m_{n-1}\right) - \psi\left(\alpha-r\right) - \psi\left(\beta+r+m_1+\cdots+m_{n-1}\right)\right]$$

$$+\left(2k_n^2\right)^{\ell-\beta} \prod_{j=1}^{n-1} \left(1-k_j^2\right)^{-\gamma_j}$$

$$\cdot \sum_{m_1=0}^{\infty} \cdots \sum_{m_{n-1}=0}^{\infty} \sum_{r=0}^{\ell+m_1+\cdots+m_{n-1}-1} \frac{(\ell+m_1+\cdots+m_{n-1}-r-1)!}{(\alpha)_{m_1+\cdots+m_{n-1}+\ell-r}}$$

$$\frac{(\beta-\ell)_r (\gamma_1)_{m_1} \cdots (\gamma_{n-1})_{m_{n-1}}}{r! \; m_1! \ldots m_{n-1}!} \left(\frac{k_n^2-1}{2k_n^2}\right)^r \left(\frac{2k_1^2}{k_1^2-1}\right)^{m_1} \cdots \left(\frac{2k_{n-1}^2}{k_{n-1}^2-1}\right)^{m_{n-1}}. \tag{4.5}$$

Let $\gamma_n = \beta + \ell, \; \ell = 1, 2, 3, \ldots$. If $m_1 + \cdots + m_{n-1} < \ell$, we have

$${}_2F_1\left(\beta+m_1+\cdots+m_{n-1}, \; \beta+\ell; \; \alpha+\beta+m_1+\cdots+m_{n-1}; \; \frac{2k_n^2}{k_n^2-1}\right)$$

$$= \frac{\Gamma\left(\alpha+\beta+m_1+\cdots+m_{n-1}\right)}{\Gamma(\beta+\ell)\Gamma(\alpha)} \left(\frac{1-k_n^2}{2k_n^2}\right)^{\beta+\ell}$$

$$\sum_{r=0}^{\infty} \frac{(\beta+m_1+\cdots+m_{n-1})_{r+\ell-m_1-\cdots-m_{n-1}} (1-\alpha)_{r+\ell-m_1-\cdots-m_{n-1}}}{r! \, (r+\ell-m_1-\cdots-m_{n-1})!} \left(\frac{k_n^2-1}{2k_n^2}\right)^r$$

$$\left[\ln\left(2k_n^2\right) - \ln\left(1-k_n^2\right) + \psi(1+r) - \psi\left(\beta+\ell+r\right)\right.$$

$$\left. +\psi\left(1+r+\ell-m_1-\cdots-m_{n-1}\right) - \psi\left(\alpha+m_1+\cdots+m_{n-1}-\ell-r\right)\right]$$

$$+\frac{\Gamma\left(\alpha+\beta+m_1+\cdots+m_{n-1}\right)}{\Gamma(\beta+\ell)} \left(\frac{1-k_n^2}{2k_n^2}\right)^{\beta+m_1+\cdots+m_{n-1}}$$

$$\sum_{r=0}^{\ell-m_1-\cdots-m_{n-1}-1} \frac{(\ell-m_1-\cdots-m_{n-1}-r-1)! \, (\beta+m_1+\cdots+m_{n-1})_r}{\Gamma(\alpha-r) \; r!} \left(\frac{k_n^2-1}{2k_n^2}\right)^r. \tag{4.6}$$

If $m_1+\cdots+m_{n-1}\geq \ell$, we have

$$
{}_2F_1\left(\beta+m_1+\cdots+m_{n-1},\ \beta+\ell;\ \alpha+\beta+m_1+\cdots+m_{n-1};\ \frac{2k_n^2}{k_n^2-1}\right)
$$

$$
=\frac{\Gamma\left(\alpha+\beta+m_1+\cdots+m_{n-1}\right)}{\Gamma\left(\beta+m_1+\cdots+m_{n-1}\right)\Gamma\left(\alpha+m_1+\cdots+m_{n-1}-\ell\right)}\left(\frac{1-k_n^2}{2k_n^2}\right)^{\beta+m_1+\cdots+m_{n-1}}
$$

$$
\sum_{r=0}^{\infty}\frac{(\beta+\ell)_{r+m_1+\cdots+m_{n-1}-\ell}\left(1+\ell-\alpha-m_1-\cdots-m_{n-1}\right)_{r+m_1+\cdots+m_{n-1}-\ell}}{r!\ \left(r+m_1+\cdots+m_{n-1}-\ell\right)!}
$$

$$
\left(\frac{k_n^2-1}{2k_n^2}\right)^r\left[\ln\left(2k_n^2\right)-\ln\left(1-k_n^2\right)+\psi\left(1+r+m_1+\cdots+m_{n-1}-\ell\right)+\psi(1+r)\right.
$$

$$
\left.-\psi\left(\beta+r+m_1+\cdots+m_{n-1}\right)-\psi\left(\alpha-r\right)\right]
$$

$$
+\frac{\Gamma\left(\alpha+\beta+m_1+\cdots+m_{n-1}\right)}{\Gamma\left(\beta+m_1+\cdots+m_{n-1}\right)}\left(\frac{1-k_n^2}{2k_n^2}\right)^{\beta+\ell}
$$

$$
\cdot\sum_{r=0}^{m_1+\cdots+m_{n-1}-\ell-1}\frac{\left(m_1+\cdots+m_{n-1}-\ell-r-1\right)!\,(\beta+\ell)_r}{\Gamma\left(\alpha+m_1+\cdots+m_{n-1}-\ell-r\right)\ r!}\left(\frac{k_n^2-1}{2k_n^2}\right)^r. \tag{4.7}
$$

Consequently,

$$
Z^{(\alpha,\beta)}_{(\gamma_1,\cdots,\gamma_{n-1},\beta+\ell)}(k)=\left(2k_n^2\right)^{-\beta-\ell}\prod_{j=1}^{n-1}\left(1-k_j^2\right)^{-\gamma_j}
$$

$$
\sum_{r=0}^{\infty}\sum_{m_1+\cdots+m_{n-1}<\ell}\frac{(1-\alpha)_{r+\ell-m_1-\cdots-m_{n-1}}\,(\beta+\ell)_r\,(\gamma_1)_{m_1}\dots(\gamma_{n-1})_{m_{n-1}}}{\left(r+\ell-m_1-\cdots-m_{n-1}\right)!\ r!\ m_1!\dots m_{n-1}!}
$$

$$
\left(\frac{k_n^2-1}{2k_n^2}\right)^r\left(\frac{2k_1^2}{k_1^2-1}\right)^{m_1}\cdots\left(\frac{2k_{n-1}^2}{k_{n-1}^2-1}\right)^{m_{n-1}}\left[\ln\left(2k_n^2\right)-\ln\left(1-k^2\right)+\psi(1+r)\right.
$$

$$
\left.+\psi\left(1+r+\ell-m_1-\cdots-m_{n-1}\right)-\psi\left(\beta+\ell+r\right)-\psi\left(\alpha+m_1+\cdots+m_{n-1}-\ell-r\right)\right]
$$

$$
+\frac{\left(2k_n^2\right)^{-\beta}}{(\beta)_\ell}\left(1-k_n^2\right)^{-\ell}\prod_{j=1}^{n-1}\left(1-k_j^2\right)^{-\gamma_j}
$$

$$
\sum_{r+m_1+\cdots+m_{n-1}<\ell}\frac{(\beta)_{r+m_1+\cdots+m_{n-1}}\,(1-\alpha)_r\,\left(\ell-m_1-\cdots-m_{n-1}-r-1\right)!(\gamma_1)_{m_1}\cdots(\gamma_{n-1})_{m_{n-1}}}{r!\,m_1!\dots m_{n-1}!}
$$

$$
\left(\frac{1-k_n^2}{2k_n^2}\right)^r\left(\frac{k_1^2\left(1-k_n^2\right)}{k_n^2\left(1-k_1^2\right)}\right)^{m_1}\cdots\left(\frac{k_{n-1}^2\left(1-k_n^2\right)}{k_n^2\left(1-k_{n-1}^2\right)}\right)^{m_{n-1}}
$$

$$+\frac{\left(2k_n^2\right)^{-\beta}}{(\beta)_\ell}\left(k_n^2-1\right)^{-\ell}\prod_{j=1}^{n-1}\left(1-k_j^2\right)^{-\gamma_j}\sum_{r=0}^{\infty}\sum_{m_1+\cdots+m_{n-1}\geq\ell}\frac{(\beta)_{r+m_1+\cdots+m_{n-1}}}{(r+m_1+\cdots+m_{n-1}-\ell)!}$$

$$\frac{(1-\alpha)_r\,(\gamma_1)_{m_1}\cdots(\gamma_{n-1})_{m_{n-1}}}{r!\,m_1!\dots m_{n-1}!}\left(\frac{k_n^2-1}{2k_n^2}\right)^r\left(\frac{k_1^2\left(k_n^2-1\right)}{k_n^2\left(k_1^2-1\right)}\right)^{m_1}\cdots\left(\frac{k_{n-1}^2\left(k_n^2-1\right)}{k_n^2\left(k_{n-1}^2-1\right)}\right)^{m_{n-1}}$$

$$\Big[\ln\left(2k^2\right)-\ln\left(1-k_n^2\right)+\psi\left(1+r+m_1+\cdots+m_{n-1}-\ell\right)+\psi\left(1+r\right)$$

$$-\psi\left(\alpha-r\right)-\psi\left(\beta+r+m_1+\cdots+m_{n-1}\right)\Big]+\left(2k_n^2\right)^{-\beta-\ell}\prod_{j=1}^{n-1}\left(1-k_j^2\right)^{-\gamma_j}$$

$$\sum_{m_1+\cdots+m_{n-1}\geq\ell}\ \sum_{r=0}^{m_1+\cdots+m_{n-1}-\ell-1}\frac{(m_1+\cdots+m_{n-1}-\ell-r-1)!}{(\alpha)_{m_1+\cdots+m_{n-1}-\ell-r}}$$

$$\frac{(\beta+\ell)_r\,(\gamma_1)_{m_1}\cdots(\gamma_{n-1})_{m_{n-1}}}{r!\,m_1!\dots m_{n-1}!}\left(\frac{k_n^2-1}{2k_n^2}\right)^r\left(\frac{2k_1^2}{k_1^2-1}\right)^{m_1}\cdots\left(\frac{2k_{n-1}^2}{k_{n-1}^2-1}\right)^{m_{n-1}}. \tag{4.8}$$

Following the same procedure one can establish asymptotic formulas for other families of elliptic-type integrals.

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