SPECTRUM OF SIGNALS

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Abstract. Finite-energy high frequency signals, band-pass frequency signals, and band-stop frequency signals are characterized.

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1. Introduction

Let $f \in L_2(R)$ and $\hat{f}$ be its Fourier transform

$$(Ff)(\xi) = \hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\xi} f(x)\, dx.$$

In signal analysis [4], f is called a finite-energy signal, and $\hat{f}$ is the signal frequency content. If the signal frequency content $\hat{f}$ vanishes a.e. outside a band of the form $[-\sigma, \sigma]$, the signal is band-limited, and the smallest such σ is called the bandwidth of the signal. Conversely, if the signal frequency content $\hat{f}$ vanishes a.e. on a band $(-\sigma, \sigma)$, the signal is called a high frequency signal. A signal is a band-pass frequency signal, if the signal frequency content vanishes a.e. outside a band $[-\sigma_2, -\sigma_1] \cup [\sigma_1, \sigma_2]$, $0 < \sigma_1 < \sigma_2 < \infty$, and a signal is a band-stop frequency signal, if the signal frequency content vanishes a.e. on a band $(-\sigma_2, -\sigma_1) \cup (\sigma_1, \sigma_2)$. They are the outputs of signals transmitting through some low-pass, high-pass, band-pass, or band-stop (or narrow-band) filters, respectively. These filters are time-invariant linear systems most commonly used in signal processing. Therefore, it is very important to characterize those classes of signals. A band-limited signal is described by the famous Paley-Wiener theorem, stating that a finite-energy signal f is band-limited if and only if f can be continued analytically into the whole complex plane as an entire function of exponential type. Another characterization of band-limited signals was discovered by Bang [1]. His result can be rephrased as follows

Theorem A [1]. *A finite-energy signal f is band-limited with bandwidth σ if and only if f is infinitely differentiable, $f^{(n)} \in L_2(R)$ for all n, and*

$$\lim_{n\to\infty} \|f^{(n)}\|_2^{1/n} = \sigma. \tag{1}$$

Boas [2] was the first to study high frequency signals. He showed that the signal frequency content $\hat{f}$ vanishes almost everywhere on the band $(-1, 1)$ if and only if

$$(B(Bf))(x) = -f(x), \qquad x \in R,$$

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where B is the Boas transform

$$(Bf)(x) = \frac{1}{\pi} \int_0^\infty \frac{f(x+t) - f(x-t)}{t^2} \sin t \, dt.$$

Boas' characterization is, however, more difficult to verify than a direct application of the definition of high frequency signals: $(Ff)(\xi) = 0, \quad \xi \in (-1, 1)$.

In this paper we shall propose another description of high frequency signals, that is easier to verify. Together with Bang's characterization of band-limited signals, it enables us to describe fully band-pass and band-stop frequency signals.

2. High frequency signals

Let

$$(If)(x) := \int_x^\infty f(y)dy,$$

the improper indefinite Riemann integral, and $I^n = (I)^n$. If $f(x)(1 + |x|)^{n-1} \in L_1(R)$, then

$$(I^n f)(x) = \int_x^\infty \frac{(y-x)^{n-1}}{(n-1)!} f(y) \, dy =: I_+^n f(x),$$

where I_+^α is the Weyl fractional integral operator of order α. But in general, $I^n f$ as an n-fold primitive of f may exist while $I_+^n f$ does not converge as an improper Riemann integral. As an example take $f(x) = (1 + ix)^{-1} e^{ix}$.

Our main result is the following

Theorem 1. *Let $f \in L_2(R)$. Then f is a high frequency signal if and only if $I^n f$ exists and belongs to $L_2(R)$ for all n, and*

$$\lim_{n \to \infty} \|I^n f\|_2^{1/n} = d < \infty. \tag{2}$$

Moreover,

$$d = \sigma_{\hat{f}}^{-1}, \tag{3}$$

where

$$\sigma_{\hat{f}} = \inf\{|\xi| : \quad \xi \in \operatorname{supp} \hat{f}\}. \tag{4}$$

By $\operatorname{supp} \hat{f}$, or the spectrum of f we denote the smallest closed set outside which the frequency content $\hat{f}$ vanishes a.e.

Proof. Necessity: Let f be a high frequency signal. Then $\sigma_{\hat{f}} > 0$, and

$$\int_x^N f(y) \, dy = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \frac{e^{iN\xi} - e^{ix\xi}}{i\xi} \hat{f}(\xi) \, d\xi.$$

Since $\hat{f}$ vanishes a.e. on $(-\sigma_{\hat{f}}, \sigma_{\hat{f}})$, then $\frac{\hat{f}(\xi)}{i\xi} \in L_1(R)$, and therefore, by the Riemann-Lebesque lemma,

$$\lim_{N \to \infty} \int_{-\infty}^\infty e^{iN\xi} \frac{\hat{f}(\xi)}{i\xi} \, d\xi = 0.$$

Consequently,

$$(If)(x) = \lim_{N\to\infty}\int_x^N f(y)\,dy = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ix\xi}\,\frac{\hat f(\xi)}{-i\xi}\,d\xi.$$

Hence, the signal If has a frequency content $\frac{\hat f(\xi)}{-i\xi} \in L_2(R)$, vanishing a.e. on $(-\sigma_{\hat f}, \sigma_{\hat f})$. Thus If is also a finite-energy high frequency signal. By induction one can show that $I^n f$ is a finite-energy high frequency signal for any n. Hence, $I^n f \in L_2(R)$ for any n, and moreover,

$$(I^n f)(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ix\xi}\,\frac{\hat f(\xi)}{(-i\xi)^n}\,d\xi. \tag{5}$$

The Parseval equality for the Fourier transform yields

$$\|I^n f\|_2^2 = \int_{-\infty}^{\infty}\frac{|\hat f(\xi)|^2}{|\xi|^{2n}}\,d\xi. \tag{6}$$

Since $\hat f$ vanishes a.e. on the interval $(-\sigma_{\hat f}, \sigma_{\hat f})$ we have

$$\int_{-\infty}^{\infty}\frac{|\hat f(\xi)|^2}{|\xi|^{2n}}\,d\xi = \int_{\sigma_{\hat f}<|\xi|<\infty}\frac{|\hat f(\xi)|^2}{|\xi|^{2n}}\,d\xi \le \frac{1}{\sigma_{\hat f}^{2n}}\int_{\sigma_{\hat f}<|\xi|<\infty}|\hat f(\xi)|^2\,d\xi$$

$$= \sigma_{\hat f}^{-2n}\int_{-\infty}^{\infty}|\hat f(\xi)|^2\,d\xi = \sigma_{\hat f}^{-2n}\|\hat f\|_2^2.$$

Consequently,

$$\limsup_{n\to\infty}\|I^n f\|_2^{1/n} \le \sigma_{\hat f}^{-1}. \tag{7}$$

On the other hand, because $\sigma_{\hat f}$ is the infimum of $|\xi|$, where ξ belongs to the support of $\hat f$, for any positive ϵ we have

$$\int_{\sigma_{\hat f}<|\xi|<\sigma_{\hat f}+\epsilon}|\hat f(\xi)|^2 d\xi > 0.$$

Hence,

$$\int_{-\infty}^{\infty}\frac{|\hat f(\xi)|^2}{|\xi|^{2n}}\,d\xi \ge \int_{\sigma_{\hat f}<|\xi|<\sigma_{\hat f}+\epsilon}\frac{|\hat f(\xi)|^2}{|\xi|^{2n}}\,d\xi \ge \frac{1}{(\sigma_{\hat f}+\epsilon)^{2n}}\int_{\sigma_{\hat f}<|\xi|<\sigma_{\hat f}+\epsilon}|\hat f(\xi)|^2\,d\xi,$$

and therefore,

$$\liminf_{n\to\infty}\|I^n f\|_2^{1/n} \ge (\sigma_{\hat f}+\epsilon)^{-1}. \tag{8}$$

Because $\epsilon > 0$ is arbitrary, the equalities (2) and (3) follow from (7) and (8).

Sufficiency: Let $I^n f$ exist and belong to $L_2(R)$ for any n and let the sequence $\|I^n f\|_2^{1/n}$ converge to $d < \infty$. By $\hat f_n$ we denote the frequency content of $I^n f$. Then $\hat f_n \in L_2(R)$ for any n. Since $\frac{d^n}{dx^n}(I^n f)(x) = (-1)^n f(x)$, we have $\hat f(\xi) = (-i\xi)^n \hat f_n(\xi)$. Hence $\frac{\hat f(\xi)}{(-i\xi)^n} = \hat f_n(\xi) \in L_2(R)$ for any n, and formulas (5) and (6) hold.

Suppose that f is not a high frequency signal. Then $\hat{f}$ does not vanish in any neighborhood of 0, and for any $\epsilon > 0$,

$$\int_{|\xi|<\epsilon} |\hat{f}(\xi)|^2 d\xi > 0.$$

Hence,

$$\|I^n f\|_2^2 = \int_{-\infty}^{\infty} \frac{|\hat{f}(\xi)|^2}{|\xi|^{2n}}\, d\xi \geq \int_{|\xi|<\epsilon} \frac{|\hat{f}(\xi)|^2}{|\xi|^{2n}}\, d\xi \geq \frac{1}{\epsilon^{2n}} \int_{|\xi|<\epsilon} |\hat{f}(\xi)|^2\, d\xi,$$

and therefore,

$$\lim_{n\to\infty} \|I^n f\|_2^{1/n} \geq \epsilon^{-1}.$$

Because $\epsilon > 0$ is arbitrary, it follows

$$\lim_{n\to\infty} \|I^n f\|_2^{1/n} = \infty,$$

that contradicts the assumption of convergence of the sequence $\|I^n f\|_2^{1/n}$. Thus, f is a high frequency signal. The necessity part of the proof then shows that the limit d equals to $\sigma_{\hat{f}}^{-1}$. $\square$

3. Band-pass and band-stop frequency signals

A signal is a band-pass frequency signal if and only if it is both a band-limited and a high frequency signal. Consequently, from Theorems A and 1 we obtain

Corollary 1. *Necessary and sufficient conditions for a finite-energy signal f to be a band-pass frequency signal are*
a) *$f^{(n)}$ exists and belongs to $L_2(R)$ for any n, and*

$$\lim_{n\to\infty} \|f^{(n)}\|_2^{1/n} = \sigma_2 < \infty.$$

b) *$I^n f$ exists and belongs to $L_2(R)$ for any n, and*

$$\lim_{n\to\infty} \|I^n f\|_2^{1/n} = \sigma_1^{-1} < \infty.$$

Moreover, the spectrum of f belongs to $[-\sigma_2, -\sigma_1] \cup [\sigma_1, \sigma_2]$.

Proof. Let f be a band-pass frequency signal, and

$$\sigma_1 := \inf\{|\xi| \,:\, \xi \in \operatorname{supp} \hat{f}\}, \quad \sigma_2 := \sup\{|\xi| \,:\, \xi \in \operatorname{supp} \hat{f}\}.$$

Then $0 < \sigma_1 \leq \sigma_2 < \infty$. Since f is a band-limited signal with the bandwidth σ_2, by Theorem A, the condition a) holds. Because f is also a high frequency signal, by Theorem 1, the condition b) is valid.

Conversely, let conditions a) and b) hold. From a) and Theorem A it follows that f is a band-limited signal with the bandwidth σ_2. From b) and Theorem 1 it follows that f is a high frequency signal, and the signal frequency content $\hat{f}$ vanishes a.e on $(-\sigma_1, \sigma_1)$.

Consequently, the spectrum of f belongs to $[-\sigma_2, -\sigma_1] \cup [\sigma_1, \sigma_2]$. $\square$

On the other hand, the signal frequency content $\hat{f}(\xi)$ vanishes a.e. on $(-\sigma_2, -\sigma_1) \cup (\sigma_1, \sigma_2)$, if and only if both $\hat{f}\left(\xi \pm \frac{\sigma_1+\sigma_2}{2}\right)$ vanish on $\left(-\frac{\sigma_2-\sigma_1}{2}, \frac{\sigma_2-\sigma_1}{2}\right)$, or equivalently, if both signals $e^{\mp\frac{(\sigma_1+\sigma_2)xi}{2}} f(x)$ are high frequency signals. Thus, we have

Corollary 2. *A finite-energy signal f is a band-stop frequency signal, and the spectrum of f does not contain the band $(-\sigma_2, -\sigma_1) \cup (\sigma_1, \sigma_2)$ if and only if $I^n\left(e^{\pm\frac{(\sigma_1+\sigma_2)xi}{2}} f(x)\right)$ exists and belongs to $L_2(R)$ for any n, and*

$$\lim_{n\to\infty} \left\| I^n\left(e^{\pm\frac{(\sigma_1+\sigma_2)xi}{2}} f(x)\right)\right\|_2^{1/n} \le \frac{2}{\sigma_2 - \sigma_1}.$$

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