A Finite and an Infinite Whittaker Integral Transform

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Abstract—This paper is concerned with the construction and study of specific finite and infinite integral transforms arising from a singular Sturm-Liouville problem on a half line, which is connected to the hydrogen atom equation. The paper goes further to describe fully the image of some spaces of square integrable functions with respect to some measure under the finite and infinite integral transforms. © 2003 Elsevier Ltd. All rights reserved.

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1. INTRODUCTION

Let \( \mathbb{R} \) denote the set of all real numbers and \( \mathbb{R}^+ = (0, \infty) \). Consider the following singular differential equation in Sturm-Liouville form:

\[
\frac{d^2y}{dt^2} + \left[ \lambda + \frac{1/4 - \mu^2}{t^2} + \frac{c}{t} \right] y = 0, \quad t \in \mathbb{R}^+,
\]

where \( c \) and \( \mu \) are real numbers. This is the celebrated equation from which physicists have been able to derive the theory of the hydrogen atom.

The change of variables \( x = -2i\sqrt{\lambda}t \) and \( \kappa = ic/2\sqrt{\lambda} \) transforms equation (1) into the Whittaker equation

\[
\frac{d^2w}{dx^2} + \left[ -\frac{1}{4} + \frac{1/4 - \mu^2}{x^2} + \frac{\kappa}{x} \right] w = 0.
\]

Two solutions of (2) are the Whittaker functions \( M_{\kappa,\mu}(x) \) and \( W_{\kappa,\mu}(x) \) [1]. We shall use the Whittaker functions \( M_{\kappa,\mu}(x) \) and \( W_{\kappa,\mu}(x) \), and apply the Sturm-Liouville eigenfunction expansion theory described in [2] to equation (1) in order to construct a Lebesgue-Stieltjes measure \( d\rho \), and an isometric integral transform from \( L_2(\mathbb{R}^+, d\rho) \) onto \( L_2(\mathbb{R}^+) \). To ensure that the expansion formula does not have a discrete part, i.e., a series expansion, we shall assume for the rest of this paper that \( c \leq 0 \) and \( \mu > 1 \). We exhibit the integral transform explicitly, call it a Whittaker
transform, and adopt it as the central part of our study. Our construction procedure starts with the beginning of Section 2 and crystallizes in Section 3. The Whittaker transform we define in Section 3 is different from the known integral transforms whose kernel is a Whittaker function and are listed in [3]. A well-known property of the Fourier transform is that $f(t)$, $tf(t) \in L^2(\mathbb{R})$ if and only if the Fourier transform $\hat{f}(s)$ and $\frac{d}{ds} \hat{f}(s)$ belong to $L^2(\mathbb{R})$. It is interesting that the Whittaker transform we construct in this paper turns out to share a similar property, which fully characterizes the image of $L^2(\mathbb{R}^+, (1 + x^2)\frac{d\rho}{dx})$ under the Whittaker transform.

Since $L^2((0, A), d\rho)$, $A > 0$, can be embedded into $L^2(\mathbb{R}^+, d\rho)$ in a natural way, the restriction of the Whittaker transform to $L^2((0, A), d\rho)$ gives rise to a finite integral transform, which we call the finite Whittaker transform. The main theme of Section 4 is the study of the finite Whittaker integral transform acting on $L^2((0, A), d\rho)$. We shall fully describe the image of $L^2((0, A), d\rho)$ under the finite Whittaker transform. In general, it is much harder to describe the image of $L^2(d\rho)$ under a finite integral transform, that is, to obtain a Paley-Wiener type theorem [4,5], than under an infinite integral transform, that is, to obtain a Plancherel type theorem [2,6,7]. The techniques used in this paper share some similarities to those of [5,8,9].

2. THE TITCHMARSH-WEYL FUNCTIONS

The Wronskian of the Whittaker functions $M_{\kappa, \mu}(x)$ and $W_{\kappa, \mu}(x)$ is [1]

$$W(M_{\kappa, \mu}(x), W_{\kappa, \mu}(x)) = -\frac{\Gamma(1 + 2\mu)}{\Gamma(1/2 + \mu - \kappa)}.$$  \hspace{1cm} (3)

Since $c \leq 0$ and $\mu > 1$, it follows that

$$M_{\kappa, \mu}(-2i\sqrt{\lambda}t) \quad \text{and} \quad W_{\kappa, \mu}(-2i\sqrt{\lambda}t), \quad \text{where} \quad \kappa = \frac{i c}{2\sqrt{\lambda}},$$

are two linearly independent solutions of (1).

Let $\varphi(t, \lambda)$ be the solution of (1) that satisfies the initial conditions

$$\varphi(1, \lambda) = 0 \quad \text{and} \quad \varphi'(1, \lambda) = -1.$$ 

Then

$$\varphi(t, \lambda) = -\frac{\Gamma(1/2 + \mu - \kappa)}{2\sqrt{\lambda}\Gamma(1 + 2\mu)} \left[ W_{\kappa, \mu}(-2i\sqrt{\lambda}) M_{\kappa, \mu}(-2i\sqrt{\lambda}t) - M_{\kappa, \mu}(-2i\sqrt{\lambda}) W_{\kappa, \mu}(-2i\sqrt{\lambda}t) \right].$$

Similarly, if $\theta(t, \lambda)$ is the solution of (1) satisfying the initial conditions

$$\theta(1, \lambda) = 1 \quad \text{and} \quad \theta'(1, \lambda) = 0,$$

then

$$\theta(t, \lambda) = -\frac{\Gamma(1/2 + \mu - \kappa)}{\Gamma(1 + 2\mu)} \left[ W_{\kappa, \mu}'(-2i\sqrt{\lambda}) M_{\kappa, \mu}(-2i\sqrt{\lambda}t) - M_{\kappa, \mu}'(-2i\sqrt{\lambda}) W_{\kappa, \mu}(-2i\sqrt{\lambda}t) \right].$$

We shall derive what is known as the Titchmarsh-Weyl functions for the problem under consideration. But since our problem is singular at 0 as well as $\infty$, we must split the analysis of the problem in two separate intervals, say $(0, 1]$ and $[1, \infty)$, each containing only one singularity. We begin with the interval $(0, 1]$.

It is known [2] that there exists a function $m_1(\lambda)$, called a Titchmarsh-Weyl function, analytic in the upper half plane such that for each $\lambda$ with $\text{Im} \lambda > 0$,

$$\Psi_1(t, \lambda) = \theta(t, \lambda) + m_1(\lambda) \varphi(t, \lambda),$$ \hspace{1cm} (4)
as a function of $t$, is in $L_2(0,1)$. Some computation leads to
\[
m_1(\lambda) = \frac{2i\sqrt{\lambda}M_{\kappa,\mu}'(-2i\sqrt{\lambda})}{M_{\kappa,\mu}(-2i\sqrt{\lambda})} \tag{5}
\]
and
\[
\Psi_1(t, \lambda) = \frac{M_{\kappa,\mu}(-2\sqrt{\lambda}t)}{M_{\kappa,\mu}(-2i\sqrt{\lambda})} \tag{6}
\]

Now we consider the problem in the interval $[1, \infty)$. Let $m_2(\lambda)$ (see [2]) be the Titchmarsh-Weyl function that is analytic in the upper half plane such that for each $\lambda$ with $\text{Im} \lambda > 0$,
\[
\Psi_2(t, \lambda) = \theta(t, \lambda) + m_2(\lambda)\varphi(t, \lambda), \tag{7}
\]
as a function of $t$, is in $L_2(1, \infty)$. A computation will yield
\[
m_2(\lambda) = \frac{2i\sqrt{\lambda}W'_{\kappa,\mu}(-2i\sqrt{\lambda})}{W_{\kappa,\mu}(-2i\sqrt{\lambda})}, \quad \text{Im} \lambda > 0 \tag{8}
\]
and
\[
\Psi_2(t, \lambda) = \frac{W_{\kappa,\mu}(-2\sqrt{\lambda}t)}{W_{\kappa,\mu}(-2i\sqrt{\lambda})}. \tag{9}
\]

3. THE WHITTAKER TRANSFORM

It is known (see [2]) that
\[
\xi(\lambda) = \frac{-1}{\pi} \lim_{\delta \to 0^+} \int_0^\lambda \text{Im} \left( \frac{1}{m_1(u + i\delta) - m_2(u + i\delta)} \right) du \tag{10}
\]
is a nondecreasing function on $\mathbb{R}$, and therefore, defines a Lebesgue-Stieltjes measure $d\xi$ on $\mathbb{R}$. If the Titchmarsh-Weyl function $m_1(\lambda)$ tends to a real limit as $\text{Im} \lambda \to 0^+$, then, following [2], for $f \in L_2(\mathbb{R}^+)$,
\[
G(\lambda) = \int_0^\infty \Psi_1(t, \lambda)f(t) \, dt, \tag{11}
\]
where $\Psi_1(x, \lambda)$ is as in (4), belongs to $L_2(\mathbb{R}, d\xi(\lambda))$, and the inverse formula
\[
f(t) = \int_{-\infty}^\infty G(\lambda)\Psi_1(t, \lambda) \, d\xi(\lambda), \tag{12}
\]
along with the Parseval identity
\[
\int_0^\infty |f(t)|^2 \, dt = \int_{-\infty}^\infty |G(\lambda)|^2 \, d\xi(\lambda), \tag{13}
\]
hold. Here, relation (11) is understood in the sense that
\[
\left\| G(\lambda) - \int_a^b \Psi_1(t, \lambda)f(t) \, dt \right\|_{L_2(\mathbb{R}, d\xi(\lambda))} \to 0, \quad \text{as } a \to 0^+ \text{ and } b \to \infty,
\]
while (12) is understood in the sense
\[
\|f(t) - \int_a^b G(\lambda) \Psi_1(t, \lambda) d\xi(\lambda)\|_{L_2(\mathbb{R}^+)} \to 0, \quad \text{as } a \to -\infty \text{ and } b \to \infty.
\]

To determine the sought after integral transform, we first observe that expression (5) clearly shows that the function \(m_1(\lambda)\) is continuously extendable to the real \(\lambda\)-axis, \(\lambda \neq 0\). We show that \(m_1(\lambda)\) is real-valued for \(\lambda\) a nonzero real number. In fact, if \(\lambda < 0\), then \(\sqrt{\lambda} = i\sqrt{|\lambda|}\) and \(\kappa = c/2\sqrt{|\lambda|}\). Thus, (5) yields
\[
\text{Im} \ m_1(\lambda) = -2\sqrt{|\lambda|} \text{Im} \ \frac{M'_{\kappa, \mu} \left(2\sqrt{|\lambda|}\right)}{M_{\kappa, \mu} \left(2\sqrt{|\lambda|}\right)} = 0, \quad \lambda < 0, \quad (14)
\]
since \(M'_{\kappa, \mu}(2\sqrt{|\lambda|})\) and \(M_{\kappa, \mu}(2\sqrt{|\lambda|})\) are real quantities.

If \(\lambda > 0\), then \(\kappa = ci/2\sqrt{\lambda}\) is purely imaginary so that
\[
\frac{M_{\kappa, \mu} \left(-2i\sqrt{\lambda}\right)}{M_{\kappa, \mu} \left(-2i\sqrt{\lambda}\right)} = M_{-\kappa, \mu} \left(2i\sqrt{\lambda}\right), \quad (15)
\]
and a computation shows that
\[
\text{Im} \ m_1(\lambda) = \frac{2\sqrt{\lambda}}{|M_{\kappa, \mu} \left(-2i\sqrt{\lambda}\right)|^2} \text{Re} \left[ M'_{\kappa, \mu} \left(-2i\sqrt{\lambda}\right) M_{-\kappa, \mu} \left(2i\sqrt{\lambda}\right) \right], \quad \lambda > 0. \quad (16)
\]
Since for any complex number \(z\), \(2 \text{Re} \ z = z + \bar{z}\), one obtains with the aid of (15),
\[
2 \text{Re} \left[ M'_{\kappa, \mu} \left(-2i\sqrt{\lambda}\right) M_{-\kappa, \mu} \left(2i\sqrt{\lambda}\right) \right] = -W(M_{\kappa, \mu}(x), M_{-\kappa, \mu}(-x)) \left(-2i\sqrt{\lambda}\right) = 0, \quad (17)
\]
where the last equality follows from the fact that \(M_{\kappa, \mu}(x)\) and \(M_{-\kappa, \mu}(-x)\) are linearly dependent solutions for the Whittaker equation [1]. Combining (14), (16), and (17) gives
\[
\text{Im} \ m_1(\lambda) = 0, \quad \text{for all real } \lambda \neq 0.
\]
Thus, relations (11)-(13) hold.

To obtain an explicit formula for \(\xi'(\lambda)\), we determine
\[
m_1(\lambda) - m_2(\lambda) = 2i\sqrt{\lambda} \left[ \frac{\frac{M'_{\kappa, \mu}}{M_{\kappa, \mu}} \left(-2i\sqrt{\lambda}\right) - \frac{W'_{\kappa, \mu}}{W_{\kappa, \mu}} \left(-2i\sqrt{\lambda}\right)}{\Gamma(1/2 + \mu - \kappa)} \right] = \frac{2i\sqrt{\lambda} \Gamma(1 + 2\mu)}{\Gamma(1/2 + \mu - \kappa)} \left[ \frac{1}{M_{\kappa, \mu} \left(-2i\sqrt{\lambda}\right) W_{\kappa, \mu} \left(-2i\sqrt{\lambda}\right)} \right],
\]
where the last equality follows from (3). Therefore,
\[
\frac{1}{m_1(\lambda) - m_2(\lambda)} = \frac{\Gamma(1/2 + \mu - \kappa)}{2i\sqrt{\lambda} \Gamma(1 + 2\mu)} M_{\kappa, \mu} \left(-2i\sqrt{\lambda}\right) W_{\kappa, \mu} \left(-2i\sqrt{\lambda}\right),
\]
which is well defined and continuously extendable to the real \(\lambda\)-axis, \(\lambda \neq 0\).

If \(\lambda < 0\), then \(\sqrt{\lambda} = i\sqrt{|\lambda|}\), and therefore, \(\kappa = ci/2\sqrt{\lambda}\), \(M_{\kappa, \mu}(-2i\sqrt{\lambda})\), and \(W_{\kappa, \mu}(-2i\sqrt{\lambda})\) are all real quantities. Consequently,
\[
\text{Im} \ \frac{1}{m_1(\lambda) - m_2(\lambda)} = 0, \quad \lambda < 0.
\]
For \( \lambda > 0 \),

\[
\text{Im} \frac{1}{m_1(\lambda) - m_2(\lambda)} = -\frac{1}{2\sqrt{\Lambda(1 + 2\mu)}} \text{Re} \left[ \Gamma \left( \frac{1}{2} + \mu - \kappa \right) M_{\kappa, \mu} \left( -2i\sqrt{\lambda} \right) W_{\kappa, \mu} \left( -2i\sqrt{\lambda} \right) \right].
\]

Thus, it immediately follows from (10) that

\[
\xi'(\lambda) = -\frac{1}{\pi} \text{Im} \frac{1}{m_1(\lambda) - m_2(\lambda)} = \begin{cases} 
\frac{1}{2\pi \sqrt{\Lambda(1 + 2\mu)}} \text{Re} \left[ \Gamma \left( \frac{1}{2} + \mu - \kappa \right) M_{\kappa, \mu} \left( -2i\sqrt{\lambda} \right) W_{\kappa, \mu} \left( -2i\sqrt{\lambda} \right) \right], & \lambda > 0, \\
0, & \lambda < 0.
\end{cases}
\]

The last formula for \( \xi'(\lambda) \) can be further simplified. Using the Kummer's relation [1],

\[
M_{\kappa, \mu}(x) = e^{i\pi(1/2+\mu)} \text{sign } x M_{-\kappa, \mu}(-x),
\]

and observing that for \( \lambda > 0 \), one can apply (15) to obtain

\[
M_{\kappa, \mu}(-2i\sqrt{\lambda}) = e^{-i\pi(1/2+\mu)} M_{\kappa, \mu}(-2i\sqrt{\lambda}) = e^{-i\pi(1/2+\mu)} \left| M_{\kappa, \mu}(-2i\sqrt{\lambda}) \right|^2,
\]

for all \( \lambda > 0 \). Hence,

\[
M_{\kappa, \mu}(-2i\sqrt{\lambda}) = \left| M_{\kappa, \mu}(-2i\sqrt{\lambda}) \right| e^{-(\pi/2)(1/2+\mu)i}, \quad \text{for all } \lambda > 0,
\]

and therefore,

\[
\xi'(\lambda) = \begin{cases} 
\left| M_{\kappa, \mu}(-2i\sqrt{\lambda}) \right| \text{Re} \left[ e^{-(\pi/2)(1/2+\mu)i} \Gamma \left( \frac{1}{2} + \mu - \kappa \right) W_{\kappa, \mu} \left( -2i\sqrt{\lambda} \right) \right], & \lambda > 0, \\
0, & \lambda < 0.
\end{cases}
\]

We are now ready to define what we shall call the Whittaker transform. To that end, let

\[
F(\lambda) = \frac{\xi'(\lambda)}{M_{\kappa, \mu}(-2i\sqrt{\lambda})} G(\lambda)
\]

and

\[
d\rho(\lambda) = \frac{\left| M_{\kappa, \mu}(-2i\sqrt{\lambda}) \right|^2}{\xi'(\lambda)} d\lambda, \quad \lambda > 0.
\]

Then \( F(\lambda) \) belongs to \( L_2(\mathbb{R}^+, d\rho(\lambda)) \), and the inverse formula (12) for \( f \in L_2(\mathbb{R}^+) \) becomes

\[
f(t) = \int_0^\infty F(\lambda) M_{\kappa, \mu}(-2i\sqrt{\lambda}t) d\lambda,
\]

and the Parseval identity reads as

\[
\int_0^\infty |f(t)|^2 dt = \int_0^\infty |F(\lambda)|^2 d\rho(\lambda).
\]

Formula (11) takes the form

\[
F(\lambda) = \frac{\xi'(\lambda)}{M_{\kappa, \mu}^2(-2i\sqrt{\lambda})} \int_0^\infty M_{\kappa, \mu}(-2i\sqrt{\lambda}t) f(t) dt.
\]
Here, relation (20) is understood in the sense that

$$\left\| f(t) - \int_a^b F(\lambda)M_{\kappa,\mu}(-2i\sqrt{\lambda}t) \, d\lambda \right\|_{L_2(\mathbb{R}^+)} \to 0, \quad \text{as } a \to 0^+ \text{ and } b \to \infty,$$

while (22) is understood in the sense that

$$\left\| F(\lambda) - \frac{\xi'(\lambda)}{M_{\kappa,\mu}^2(-2i\sqrt{\lambda})} \int_a^b M_{\kappa,\mu}(-2i\sqrt{\lambda}t) f(t) \, dt \right\|_{L_2(\mathbb{R}^+, d\rho(\lambda))} \to 0,$$

as $a \to 0^+$ and $b \to \infty$.

We shall call a function $f \in L_2(\mathbb{R}^+)$ a Whittaker transform of $F \in L_2(\mathbb{R}^+, d\rho)$ if relation (20) holds. It is to be noted that (20) holds if and only if (22) holds. Hence, the Whittaker transform $F(\lambda) \to f(t)$ is an isometry from $L_2(\mathbb{R}^+, d\rho)$ onto $L_2(\mathbb{R}^+)$. Let

$$L = \frac{d^2}{dt^2} - q(t) = \frac{d^2}{dt^2} - \left( \frac{\mu^2 - 1/4}{t^2} - \frac{c}{t} \right). \tag{23}$$

Then the Sturm-Liouville problem (1) becomes

$$(L + \lambda)y = 0,$$

and its solutions are the eigenfunctions corresponding to the eigenvalue $\lambda$. Thus, the procedure to arrive at the Whittaker integral transform, and developed in this and the previous section, was in spirit an eigenfunction expansion related to the self-adjoint, singular, second-order differential operator $L$.

A well-known property of the Fourier transform is that $f(t),(tf(t) \in L_2(\mathbb{R})$ if and only if the Fourier transform $\hat{f}(s)$ and $\frac{d}{ds} \hat{f}(s)$ belong to $L_2(\mathbb{R})$. It is interesting that the Whittaker transform has a similar property, which we state precisely in Theorem 1. We need, however, some preparatory work before arriving at Theorem 1.

**Lemma 1.** Suppose that $f, Lf \in L_2(\mathbb{R}^+)$. Then

(a) $\lim_{t \to 0^+} f(t)M_{\kappa,\mu}^2(-2i\sqrt{\lambda}t) = 0 = \lim_{t \to 0^+} f'(t)M_{\kappa,\mu}(-2i\sqrt{\lambda}t);$

(b) $\lim_{t \to \infty} f(t)M_{\kappa,\mu}(-2i\sqrt{\lambda}t) = 0 = \lim_{t \to \infty} f'(t)M_{\kappa,\mu}(-2i\sqrt{\lambda}t).$

**Proof.** Recall that

$$q(t) = \frac{\mu^2 - 1/4}{t^2} - \frac{c}{t}.$$

Thus, the function $t^2q(t)$ is bounded on $(0, 1]$, so $t^2q(t)f(t)$ belongs to $L_2(0, 1)$ since $f$ does. Since $t^2(Lf)(t) \in L_2(0, 1)$, and

$$t^2f''(t) = t^2(Lf)(t) + t^2q(t)f(t),$$

it follows that $t^2f''(t) \in L_2(0, 1)$. Let $g$ be an extension of $f$ from $(0, 1]$ over to $(0, \infty)$ that is twice continuously differentiable on $(1, \infty)$, and $g(t) = 0$ for $t > 2$. Then $g' = f'$, $g'' = f''$ over $(0, 1)$, and $t^2g''(t) \in L_2(0, 2)$. The Hardy inequality [10,11]

$$\left\{ \int_0^\infty |h(t)|^p t^{\varepsilon-p} \, dt \right\}^{1/p} \leq \frac{p}{|\varepsilon-p+1|} \left\{ \int_0^\infty |h'(t)|^p t^{\varepsilon} \, dt \right\}^{1/p},$$

applied to $h(t) = g'(t)$, with $\varepsilon = 4$, and $p = 2$ gives

$$\int_0^2 |tg'(t)|^2 \, dt \leq \frac{4}{9} \int_0^2 |t^2g''(t)|^2 \, dt < \infty,$$
and therefore,
\[
\int_0^1 |tf'(t)|^2 \, dt = \int_0^1 |tg'(t)|^2 \, dt \leq \int_0^2 |tg'(t)|^2 \, dt < \infty.
\]
That is, \(tf'(t) \in L_2(0,1)\). But then \(tf(t)f'(t)\), and \(t^3f'(t)f''(t)\) would belong to \(L_1(0,1)\). Specializing the identity
\[
\frac{d}{dx} \left[x^k y^2(x)\right] = 2x^k y(x)y'(x) + kx^{k-1} y^2(x), \quad \text{for any } k \in \mathbb{R}, \quad \text{and} \quad x > 0,
\]
to the pairs \(k = 1, g = f\), and \(k = 3, g = f'\) we see upon integrating over the interval \([t,1]\); \(0 < t < 1\), that
\[
2 \int_t^1 xf(x)f'(x) \, dx + \int_t^1 f^2(x) \, dx = f^2(1) - tf^2(t),
\]
\[
2 \int_t^1 x^3f'(x)f''(x) \, dx + 3 \int_t^1 [xf'(x)]^2 \, dx = [f'(1)]^2 - t^3 [f'(t)]^2.
\]
Therefore, \(\lim_{t \to 0^+} tf^2(t)\), and \(\lim_{t \to 0^+} t^3[f'(t)]^2\) exist since the left-hand side of each of the above equalities has a limit as \(t \to 0^+\). Consequently, \(\lim_{t \to 0^+} t^{1/2}|f(t)|\), and \(\lim_{t \to 0^+} t^{3/2}|f'(t)|\) exist, which in turn gives
\[
\lim_{t \to 0^+} t^{-1/2}|f(t)| = 0 \quad \text{and} \quad \lim_{t \to 0^+} t^{\mu+1/2}|f'(t)| = 0, \quad \mu > 1. \quad (24)
\]
We have (see [1])
\[
M_{\kappa,\mu}\left(-2i\sqrt{\lambda}t\right) = O\left(t^{1/2+\mu}\right), \quad \text{as } t \to 0, \quad (25)
\]
and since [1, formula 2.4.9]
\[
M'_{\kappa,\mu}(x) = \frac{2\mu}{\sqrt{x}}M_{\kappa+1/2,\mu-1/2}(x) - \frac{2\mu - 1 - x}{2x}M_{\kappa,\mu}(x), \quad (26)
\]
we get, with the help of (25),
\[
M'_{\kappa,\mu}\left(-2i\sqrt{\lambda}t\right) = O\left(t^{\mu-1/2}\right), \quad \text{as } t \to 0.
\]
It thus follows that
\[
f(t)M_{\kappa,\mu}'\left(-2i\sqrt{\lambda}t\right) = O\left(t^{\mu-1/2}|f(t)|\right),
\]
\[
f'(t)M_{\kappa,\mu}\left(-2i\sqrt{\lambda}t\right) = O\left(t^{\mu+1/2}|f'(t)|\right),
\]
which along with (24) establish part (a) of the lemma.

We now turn our attention to the behavior of \(f(t)\) and \(f'(t)\) at \(\infty\). The function \(q\) is bounded on \((1,\infty)\), and therefore, \(f'' = qf + Lf\) belongs to \(L_2(1,\infty)\). We have, by a Hardy-Littlewood inequality [10, formula 259, p. 187],
\[
\int_1^\infty |f'(x)|^2 \, dx \leq 2 \left(\int_1^\infty |f(x)|^2 \, dx\right)^{1/2} \left(\int_1^\infty |f''(x)|^2 \, dx\right)^{1/2}.
\]
Thus, \(f' \in L_2(1,\infty)\) since \(f, f'' \in L_2(1,\infty)\), and therefore, \(ff' \in L_1(1,\infty)\). Now
\[
2 \int_1^t f(x)f'(x) \, dx = \int_1^t \frac{d}{dx} f^2(x) \, dx = f^2(t) - f^2(1),
\]
and the limit of the most left side exists as \( t \to \infty \). Consequently, \( \lim_{t \to \infty} f^2(t) \) exists. But \( f^2 \in L_1(1, \infty) \), so this limit must be zero. Hence,
\[
\lim_{t \to \infty} f(t) = 0.
\]

Similarly, the relation
\[
2 \int_1^t f'(x)f''(x) \, dx = \int_1^t \frac{d}{dx} \left[ f'(x) \right]^2 \, dx = \left[ f'(t) \right]^2 - \left[ f'(1) \right]^2,
\]
and the fact that \( f'f'' \), \( f^2 \in L_1(1, \infty) \) show that
\[
\lim_{t \to \infty} f'(t) = 0.
\]

From [1, formula 4.1.21],
\[
M_{\mu, \nu}(x) = \frac{x^{-\frac{\nu}{2}} e^{-x/2}}{\Gamma(1/2 + \mu + \kappa)} \frac{2F_0 \left( \frac{1}{2} + \mu + \kappa, \frac{1}{2} - \mu + \kappa; \frac{1}{x} \right)}{2F_0 \left( \frac{1}{2} + \mu - \kappa, \frac{1}{2} - \mu - \kappa; \frac{1}{x} \right)},
\]
for \(-3\pi/2 < \arg x < \pi/2\) as \(|x| \to \infty\), we obtain
\[
M_{\mu, \nu}(-2i\sqrt{\lambda}t) = \left( -2i\sqrt{\lambda}t \right)^{-\frac{\nu}{2}} e^{-i\sqrt{\lambda}t} \left( 1 + O\left( \frac{1}{t} \right) \right)
\]
\[
\quad + \frac{\left( -2i\sqrt{\lambda}t \right)^\kappa e^{2i\sqrt{\lambda}t} e^{\pi i(\nu-\mu-1/2)}}{\Gamma(1/2 + \mu + \kappa)} \left( 1 + O\left( \frac{1}{t} \right) \right) = O(1),
\]
as \( t \to \infty \). Differentiating (27), which is permissible in any closed cone \(-3\pi/2 < \alpha \leq \arg x \leq \beta < \pi/2\), we have
\[
M'_{\mu, \nu}(-2i\sqrt{\lambda}t) = O\left( \frac{1}{t} \right), \quad \text{as } t \to \infty.
\]

Since \( M_{\mu, \nu}(-2i\sqrt{\lambda}t) \) and \( M'_{\mu, \nu}(-2i\sqrt{\lambda}t) \) are bounded for \( t \) sufficiently large, and \( \lim_{t \to \infty} f(t) = \lim_{t \to \infty} f'(t) = 0 \), part (b) of the lemma follows, and completes the proof. \( \blacksquare \)

**Remark 1.** The proof of Lemma 1 shows that if \( f, Lf \in L_2(\mathbb{R}^+) \), then necessarily \( \lim_{t \to \infty} f(t) = \lim_{t \to \infty} f'(t) = 0 \).

Associated with the operator \( L \), given in (23), is the Green function \( G(t, x, \sigma) \), which is defined for a nonreal complex number \( \sigma \) with \( \text{Im} \sigma > 0 \) as
\[
G(t, x, \sigma) = \begin{cases} 
\frac{1}{\omega(\sigma)} \Psi_1(x, \sigma) \Psi_2(t, \sigma), & x \leq t, \\
\frac{1}{\omega(\sigma)} \Psi_1(t, \sigma) \Psi_2(x, \sigma), & x > t,
\end{cases}
\]
where \( \Psi_1 \) and \( \Psi_2 \) are as in (6) and (9), respectively, and
\[
\omega(\sigma) = W(\Psi_1(t, \sigma), \Psi_2(t, \sigma)) = m_1(\sigma) - m_2(\sigma).
\]

The resolvent of \( f \) is the function \( R_\sigma f \) given by
\[
(R_\sigma f)(t) = \int_0^\infty G(t, x, \sigma) f(x) \, dx
\]
\[
= \frac{\Psi_2(t, \sigma)}{\omega(\sigma)} \int_0^t \Psi_1(x, \sigma) f(x) \, dx + \frac{\Psi_1(t, \sigma)}{\omega(\sigma)} \int_t^\infty \Psi_2(x, \sigma) f(x) \, dx.
\]

(28)
It is well known [2,6] that if $f \in L_2(\mathbb{R}^+)$, then $R_\sigma f \in L_2(\mathbb{R}^+)$. Moreover, $R_\sigma f$ is twice differentiable, and

$$(L + \sigma)R_\sigma f = f.$$  

(29)

The resolvent function also has the following integral representation [6]:

$$(R_\sigma f)(t) = \int_0^\infty \Psi_1(t, \lambda) \frac{G(\lambda)}{\sigma - \lambda} d\xi(\lambda).$$

(30)

It follows from (6), (19), and (30) that

$$(R_\sigma f)(t) = \int_0^\infty \Psi_1(t, \lambda) \frac{G(\lambda)\xi'(\lambda)}{\sigma - \lambda} d\lambda = \int_0^\infty M_{\kappa, \mu} \left(-2iv\lambda t\right) \frac{F(\lambda)}{\sigma - \lambda} d\lambda,$$

(31)

for any nonreal $\sigma$ with $\Im \sigma > 0$.

**Theorem 1.** Let $L$ denote the differential operator as in (23). Then a function $f$ is the Whittaker transform of a function $F(\lambda)$ such that $F(\lambda), \lambda F(\lambda) \in L_2(\mathbb{R}^+, d\rho(\lambda))$ if and only if $f \in L_2(\mathbb{R}^+)$ and $Lf$ exists and belongs to $L_2(\mathbb{R}^+)$.\n
**Proof.** We begin with showing the "only if" part of the statement. Suppose that $f$ is the Whittaker transform of a function $F(\lambda)$ such that $F(\lambda), \lambda F(\lambda) \in L_2(\mathbb{R}^+, d\rho(\lambda))$. Let $f_1(t)$ be the Whittaker transform of $\lambda F(\lambda)$, which exists since $\lambda F(\lambda) \in L_2(\mathbb{R}^+, d\rho(\lambda))$. Then necessarily $f, f_1 \in L_2(\mathbb{R}^+)$. Fix a nonreal complex number $\sigma$ with $\Im \sigma > 0$. The integral representation (31) yields

$$f(t) = \int_0^\infty M_{\kappa, \mu} \left(-2itr\lambda t\right) F(\lambda) d\lambda = \int_0^\infty (\sigma - \lambda)M_{\kappa, \mu} \left(-2itr\lambda t\right) \frac{F(\lambda)}{\sigma - \lambda} d\lambda$$

$$= \sigma \int_0^\infty M_{\kappa, \mu} \left(-2itr\lambda t\right) F(\lambda) d\lambda - \int_0^\infty M_{\kappa, \mu} \left(-2itr\lambda t\right) \frac{\lambda F(\lambda)}{\sigma - \lambda} d\lambda$$

$$= \sigma (R_\sigma f)(t) - (R_\sigma f_1)(t) = R_\sigma (\sigma f - f_1)(t).$$

Because $\sigma f - f_1 \in L_2(\mathbb{R}^+)$, $R_\sigma (\sigma f - f_1)$ is twice differentiable and so is $f$. Moreover, formula (29) gives

$$(L + \sigma)f = (L + \sigma)R_\sigma (\sigma f - f_1) = \sigma f - f_1.$$\n
Thus, $Lf = -f_1$, and therefore, $Lf$ belongs to $L_2(\mathbb{R}^+)$. This establishes the "only if" part of the statement. We have shown, in addition, that

$$\langle Lf \rangle(t) = -\int_0^\infty \lambda F(\lambda)M_{\kappa, \mu} \left(-2itr\lambda t\right) d\lambda.$$\n
We now prove the "if" part of the statement. So assume that $f \in L_2(\mathbb{R}^+)$ and $Lf$ is well defined and belongs also to $L_2(\mathbb{R}^+)$. Since $f, Lf \in L_2(\mathbb{R}^+)$, let $F(\lambda)$ and $H(\lambda)$ be the unique functions in $L_2(\mathbb{R}^+, d\rho(\lambda))$ such that $f$ and $Lf$ are the Whittaker transforms of $F(\lambda)$ and $H(\lambda)$, respectively. Moreover, as defined by (22),

$$F(\lambda) = \frac{\xi'(\lambda)}{M_{\kappa, \mu}^2 \left(-2itr\lambda t\right)} \int_0^\infty M_{\kappa, \mu} \left(-2itr\lambda t\right) f(t) dt,$$

(32)

$$H(\lambda) = \frac{\xi'(\lambda)}{M_{\kappa, \mu}^2 \left(-2itr\lambda t\right)} \int_0^\infty M_{\kappa, \mu} \left(-2itr\lambda t\right) (Lf)(t) dt.$$  

(33)

We show that $H(\lambda) = -\lambda F(\lambda)$.\n
Choose sequences \( \{a_n\} \) and \( \{b_n\} \) of real numbers such that \( 0 < a_n < b_n < \infty \), \( a_n \to 0 \), \( b_n \to \infty \), as \( n \to \infty \), and

\[
F(\lambda) = \lim_{n \to \infty} \frac{\xi' (\lambda)}{M_{\kappa, \mu}^2 (\lambda)} \int_{a_n}^{b_n} M_{\kappa, \mu} \left( -2i \sqrt{\lambda t} \right) f(t) \, dt,
\]

(34)

\[
H(\lambda) = \lim_{n \to \infty} \frac{\xi' (\lambda)}{M_{\kappa, \mu}^2 (\lambda)} \int_{a_n}^{b_n} M_{\kappa, \mu} \left( -2i \sqrt{\lambda t} \right) (Lf)(t) \, dt,
\]

(35)
a almost everywhere. This is possible since the integrals in (32) and (33) converge in \( L_2(\mathbb{R}^+, d\rho) \)-norm, which guarantee the existence of a subsequence that converges almost everywhere as in (34) and (35). Since \((Lf)(t) = f''(t) - q(t)f(t)\),

\[
\int_{a_n}^{b_n} M_{\kappa, \mu} \left( -2i \sqrt{\lambda t} \right) (Lf)(t) \, dt = \int_{a_n}^{b_n} f''(t)M_{\kappa, \mu} \left( -2i \sqrt{\lambda t} \right) dt - \int_{a_n}^{b_n} q(t)f(t)M_{\kappa, \mu} \left( -2i \sqrt{\lambda t} \right) dt.
\]

(36)

Apply integration by parts successively to obtain

\[
\int_{a_n}^{b_n} f''(t)M_{\kappa, \mu} \left( -2i \sqrt{\lambda t} \right) dt = f'(b_n)M_{\kappa, \mu} \left( -2i \sqrt{\lambda b_n} \right) - f'(a_n)M_{\kappa, \mu} \left( -2i \sqrt{\lambda a_n} \right)
\]

(37)

\[
- \int_{a_n}^{b_n} f'(t) \frac{d}{dt} M_{\kappa, \mu} \left( -2i \sqrt{\lambda t} \right) dt
\]

\[
= Z(\lambda)(b_n) - Z(\lambda)(a_n) + \int_{a_n}^{b_n} f(t) \frac{d^2}{dt^2} M_{\kappa, \mu} \left( -2i \sqrt{\lambda t} \right) dt,
\]

where

\[
Z(\lambda) = 2i \sqrt{\lambda} f(t)M_{\kappa, \mu} \left( -2i \sqrt{\lambda t} \right) + f'(t)M_{\kappa, \mu} \left( -2i \sqrt{\lambda t} \right), \quad 0 < t < \infty.
\]

Substituting (37) for \( \int_{a_n}^{b_n} f''(t)M_{\kappa, \mu} \left( -2i \sqrt{\lambda t} \right) dt \) in (36), then combining the two integrals and recognizing that

\[
\frac{d^2}{dt^2} M_{\kappa, \mu} \left( -2i \sqrt{\lambda t} \right) - q(t)M_{\kappa, \mu} \left( -2i \sqrt{\lambda t} \right) = I. \left[ M_{\kappa, \mu} \left( -2i \sqrt{\lambda t} \right) \right] = -\lambda M_{\kappa, \mu} \left( -2i \sqrt{\lambda t} \right),
\]

we arrive at

\[
\int_{a_n}^{b_n} M_{\kappa, \mu} \left( -2i \sqrt{\lambda t} \right) (Lf)(t) \, dt = Z(\lambda)(b_n) - Z(\lambda)(a_n) - \lambda \int_{a_n}^{b_n} f(t)M_{\kappa, \mu} \left( -2i \sqrt{\lambda t} \right) dt.
\]

(38)

By Lemma 1, \( \lim_{n \to \infty} Z(\lambda)(a_n) = 0 = \lim_{n \to \infty} Z(\lambda)(b_n) \), and because of (34) and (35), letting \( n \to \infty \) in (38) yields

\[
H(\lambda) = -\lambda F(\lambda).
\]

Therefore, \( \lambda F(\lambda) \) belongs to \( L_2(\mathbb{R}^+, d\rho(\lambda)) \) since \( H(\lambda) \) does. The theorem is proved. \( \blacksquare \)

**Remark 2.** The proof of Theorem 1 shows that if \( f \) is the Whittaker transform of a function \( F(\lambda) \) such that \( F(\lambda), \lambda F(\lambda) \in L_2(\mathbb{R}^+, d\rho(\lambda)) \), then

\[
(Lf)(t) = \int_0^\infty (-\lambda) F(\lambda)M_{\kappa, \mu} \left( -2i \sqrt{\lambda t} \right) \, d\lambda,
\]

that is, \( Lf \) is the Whittaker transform of \(-\lambda F(\lambda)\).

**Remark 3.** Since \( F(\lambda), \lambda F(\lambda) \in L_2(\mathbb{R}^+, d\rho(\lambda)) \) if and only if \( F(\lambda) \) belongs to \( L_2(\mathbb{R}^+, (1 + \lambda^2) d\rho(\lambda)) \), Theorem 1 gives a description of the image of the weighted space \( L_2(\mathbb{R}^+, (1 + \lambda^2) d\rho(\lambda)) \) under the Whittaker transform (20).

The following corollary of Theorem 1 is crucial in proving the main result of Section 4.
COROLLARY 1. Let $L$ denote the differential operator as in (23). Then a function $f$ is the Whittaker transform of a function $F(\lambda)$ such that $\lambda^n F(\lambda) \in L_2(\mathbb{R}^+, \varphi(\lambda))$ for any $n = 0, 1, 2, \ldots$, if and only if $L^n f$ exists and belongs to $L_2(\mathbb{R}^+)$ for any $n = 0, 1, 2, \ldots$.

PROOF. Suppose that the function $f$ is the Whittaker transform of a function $F(\lambda)$ such that $\lambda^n F(\lambda) \in L_2(\mathbb{R}^+, \varphi(\lambda))$ for any $n = 0, 1, 2, \ldots$. Set $f_0 = f$, and $F_0(\lambda) = F(\lambda)$. Let $f_n (n = 0, 1, 2, \ldots)$ be the Whittaker transform of $F_n(\lambda) = (-\lambda)^n F(\lambda)$, which exists since $F_n(\lambda) \in L_2(\mathbb{R}^+, \varphi(\lambda))$, and moreover, $f_n \in L_2(\mathbb{R}^+)$. Since $\lambda F_n(\lambda) = (-\lambda)^{n+1} F(\lambda) \in L_2(\mathbb{R}^+, \varphi(\lambda))$, Theorem 1 would apply for $f_n$. Therefore, $L f_n$ exists and belongs to $L_2(\mathbb{R}^+)$. Moreover,

\[ (L f_n)(t) = - \int_0^\infty \lambda (-\lambda)^n F(\lambda) M_{\kappa, \mu} (-2i\sqrt{\lambda}t) \, d\lambda = f_{n+1}. \]

By iterating (39) and recognizing that $f = f_0$ and $L^0 f = f$, we obtain

\[ L^n f = f_n. \]

Thus, $L^n f$ belongs to $L_2(\mathbb{R}^+)$ since $f_n$ does.

Conversely, assume that $f$ is a function for which $L^n f \in L_2(\mathbb{R}^+)$ for any $n = 0, 1, 2, \ldots$. Then Theorem 1 would apply in relation to the function $L^n f$. Thus, $L^n f$ is the Whittaker transform of a function $F_n(\lambda)$ such that $F_n(\lambda), \lambda F_n(\lambda) \in L_2(\mathbb{R}^+, \varphi(\lambda))$. Moreover,

\[ (L^{n+1} f)(t) = L ((L^n f))(t) = \int_0^\infty (-\lambda) F_n(\lambda) M_{\kappa, \mu} (-2i\sqrt{\lambda}t) \, d\lambda. \]

That is, $L^{n+1} f$ is the Whittaker transform of $(-\lambda) F_n(\lambda)$, and therefore, we have

\[ F_{n+1}(\lambda) = (-\lambda) F_n(\lambda), \quad n = 0, 1, 2, \ldots \]

Put $F(\lambda) = F_0(\lambda)$. Then $f$ is the Whittaker transform of $F(\lambda)$, and iterating (40) we get

\[ (-\lambda)^n F(\lambda) = F_n(\lambda) \in L_2(\mathbb{R}^+, \varphi(\lambda)), \quad n = 0, 1, 2, \ldots \]

as required. \[ \square \]

REMARK 4. The proof shows that if $f$ is the Whittaker transform of a function $F(\lambda)$ such that $F(\lambda), \lambda^n F(\lambda) \in L_2(\mathbb{R}^+, \varphi(\lambda))$, then

\[ (L^n f)(t) = \int_0^\infty (-\lambda)^n F(\lambda) M_{\kappa, \mu} (-2i\sqrt{\lambda}t) \, d\lambda, \]

that is, $L^n f$ is the Whittaker transform of $(-\lambda)^n F(\lambda)$.

4. THE FINITE WHITTAKER TRANSFORM

Let $A$ be a fixed positive real number, but otherwise is arbitrary. For $F \in L_2((0, A), \varphi)$, define

\[ f_A(t) = \int_0^A F(\lambda) M_{\kappa, \mu} (-2i\sqrt{\lambda}t) \, d\lambda, \quad 0 < t < \infty. \]

We will call $f_A$ the finite Whittaker transform of $F$. Clearly, $f_A \in L_2(\mathbb{R}^+)$ and the Parseval identity (21) takes the form

\[ \int_0^\infty |f_A(t)|^2 \, dt = \int_0^A |F(\lambda)|^2 \, d\lambda. \]

This section is devoted to the description of the image of $L_2((0, A), \varphi)$ under the transform (41). For this purpose we need the following lemma.
LEMMA 2. Let $\lambda^n F(\lambda) \in L^2(\mathbb{R}^+, d\rho(\lambda))$ for any $n = 0, 1, 2, \ldots$. Then

$$\lim_{n \to \infty} \left\{ \int_0^\infty \lambda^{2n} |F(\lambda)|^2 \, d\rho(\lambda) \right\}^{1/2n} = \sup_{\lambda \in \text{supp } F} \lambda.$$

PROOF. The lemma is trivial if $F(\lambda) = 0$. Thus, suppose that $F(\lambda)$ has a compact support: $\sup_{\lambda \in \text{supp } F} \lambda = A > 0$. Then

$$\int_0^\infty \lambda^{2n} |F(\lambda)|^2 \, d\rho(\lambda) = \int_0^A \lambda^{2n} |F(\lambda)|^2 \, d\rho(\lambda) \leq A^{2n} \int_0^A |F(\lambda)|^2 \, d\rho(\lambda).$$

Hence,

$$\limsup_{n \to \infty} \left\{ \int_0^\infty \lambda^{2n} |F(\lambda)|^2 \, d\rho(\lambda) \right\}^{1/2n} \leq A \limsup_{n \to \infty} \left\{ \int_0^A |F(\lambda)|^2 \, d\rho(\lambda) \right\}^{1/2n} = A.$$

On the other hand, if $0 < \varepsilon < A = \sup_{\lambda \in \text{supp } F} \lambda$, then

$$\int_{A-\varepsilon}^A |F(\lambda)|^2 \, d\rho(\lambda) > 0.$$

Thus,

$$\liminf_{n \to \infty} \left\{ \int_0^\infty \lambda^{2n} |F(\lambda)|^2 \, d\rho(\lambda) \right\}^{1/2n} \geq \liminf_{n \to \infty} \left\{ \int_{A-\varepsilon}^A \lambda^{2n} |F(\lambda)|^2 \, d\rho(\lambda) \right\}^{1/2n} \geq \liminf_{n \to \infty} \left\{ (A-\varepsilon)^{2n} \int_{A-\varepsilon}^A |F(\lambda)|^2 \, d\rho(\lambda) \right\}^{1/2n} \geq A - \varepsilon.$$

Because $\varepsilon > 0$ is arbitrary,

$$\lim_{n \to \infty} \left\{ \int_0^\infty \lambda^{2n} |F(\lambda)|^2 \, d\rho(\lambda) \right\}^{1/2n} = A.$$

Suppose that $F$ has a unbounded support. Then for any $N$ large enough,

$$\int_N^\infty |F(\lambda)|^2 \, d\rho(\lambda) > 0.$$

Consequently,

$$\liminf_{n \to \infty} \left\{ \int_0^\infty \lambda^{2n} |F(\lambda)|^2 \, d\rho(\lambda) \right\}^{1/2n} \geq \liminf_{n \to \infty} \left\{ \int_N^\infty \lambda^{2n} |F(\lambda)|^2 \, d\rho(\lambda) \right\}^{1/2n} \geq \liminf_{n \to \infty} \left\{ N^{2n} \int_N^\infty |F(\lambda)|^2 \, d\rho(\lambda) \right\}^{1/2n} = N.$$

Letting $N \to \infty$, we obtain

$$\lim_{n \to \infty} \left\{ \int_0^\infty \lambda^{2n} |F(\lambda)|^2 \, d\rho(\lambda) \right\}^{1/2n} = \infty.$$ 

This completes the proof.

We are now in a position to state and prove the main result of this section, which describes the image of $L_2((0, A), d\rho)$ under the finite Whittaker transform.
THEOREM 2. A function $f_A$ is the finite Whittaker transform (41) of a function $F \in L_2((0, A), d\rho)$ if and only if

(i) $L^n f_A$ exists and belongs to $L_2(\mathbb{R}^+)$ for any $n = 0, 1, 2, \ldots$,
(ii) $\lim_{n \to \infty} \|L^n f_A\|_{L_2(\mathbb{R}^+)}^{1/n} \leq A$.

PROOF. We start with proving the "only if" part.

Let $F \in L_2((0, A), d\rho)$. Then its extension by 0 on $(A, \infty)$ yields a function $\tilde{F} \in L_2(\mathbb{R}^+, d\rho)$ such that $\lambda^n \tilde{F}(\lambda) \in L_2(\mathbb{R}^+, d\rho(\lambda))$ for all $n$, supp $\tilde{F} \subset (0, A)$, and the Whittaker transform of $\tilde{F}$ is again $f_A$. Thus, item (i) holds for all $n$, by virtue of Corollary 1. Since

$$ (L^n f_A)(t) = \int_0^\infty (-\lambda)^n \tilde{F}(\lambda) M_{K, \mu} \left( -2i\sqrt{\lambda} t \right) d\lambda, $$

the Parseval identity (21) yields

$$ \|L^n f_A\|_{L_2(\mathbb{R}^+)} = \left\{ \int_0^\infty \lambda^{2n} |\tilde{F}(\lambda)|^2 d\rho(\lambda) \right\}^{1/2}. \quad (42) $$

Therefore, Lemma 2 gives

$$ \lim_{n \to \infty} \|L^n f_A\|_{L_2(\mathbb{R}^+)}^{1/n} = \lim_{n \to \infty} \left\{ \int_0^\infty \lambda^{2n} |\tilde{F}(\lambda)|^2 d\rho(\lambda) \right\}^{1/2n} = \sup_{\lambda \in \text{supp} \tilde{F}} \lambda \leq A, $$

which is condition (ii). This completes the proof of the "only if" part.

We prove now the "if" part of the statement. So assume that conditions (i) and (ii) hold. Then because of (i), Corollary 1 applies, and therefore, $f_A$ is the Whittaker transform (20) of a function $F$ such that $\lambda^n \tilde{F}(\lambda) \in L_2(\mathbb{R}^+, d\rho(\lambda))$ for all $n$. Moreover, the Whittaker transform of $(-\lambda)^n \tilde{F}(\lambda)$ is $L^n f_A$, and the Parseval identity (42) holds. Since $\lambda^n \tilde{F}(\lambda) \in L_2(\mathbb{R}^+, d\rho(\lambda))$ for all $n$, one can apply Lemma 2 to obtain

$$ \lim_{n \to \infty} \|L^n f_A\|_{L_2(\mathbb{R}^+)}^{1/n} = \lim_{n \to \infty} \left\{ \int_0^\infty \lambda^{2n} |\tilde{F}(\lambda)|^2 d\rho(\lambda) \right\}^{1/2n} = \sup_{\lambda \in \text{supp} \tilde{F}} \lambda \leq A. $$

Thus, supp $\tilde{F} \subset (0, A)$ and the Whittaker transform (20) turns out to be the finite Whittaker transform (41) of $F = \tilde{F}|_{(0, A)}$, the restriction of $\tilde{F}$ to $(0, A)$.

The theorem is proved. \[\blacksquare\]

REFERENCES