Question 1. Consider the following truth table.

(a) Construct a Boolean expression having this table as its truth table. Can you simplify this expression? (2 marks)

(b) Construct a circuit having the given table as its input/output table. (2 marks)

<table>
<thead>
<tr>
<th>P</th>
<th>Q</th>
<th>R</th>
<th>S</th>
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</thead>
<tbody>
<tr>
<td>1</td>
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</tr>
</tbody>
</table>

Solution: (a) \((P \land \sim Q \land R) \lor (P \land \sim Q \land \sim R) \lor (\sim P \land Q \land \sim R)\). Note that this can be simplified to \((P \land \sim Q) \lor (\sim P \land Q \land \sim R)\).

(b)

![Logic Circuit Diagram]

Question 2. The Peirce arrow \(\downarrow\) is a logical binary operation which is defined as follows:

\[ p \downarrow q \equiv (p \lor q). \]
(a) Prove that $\sim p \equiv p \downarrow p$. (1 marks)

(b) Prove that $p \land q \equiv (p \downarrow p) \downarrow (q \downarrow q)$. (2 marks)

(c) Write $p \rightarrow q$ using Peirce arrows only. (2 marks)

**Solution:** (a) By definition $p \downarrow p \equiv \sim (p \lor p) \equiv \sim p$.

(b)

\[
(p \downarrow p) \downarrow (q \downarrow q) \equiv (\sim p) \downarrow (\sim q) \quad \text{By Part (a)}
\]

\[
\equiv \sim (p \lor \sim q) \quad \text{By Definition}
\]

\[
\equiv (\sim (\sim p)) \land (\sim (\sim q)) \quad \text{De Morgan’s law}
\]

\[
\equiv p \land q
\]

(c)

\[
p \rightarrow q \equiv \sim p \lor q \quad \text{By a theorem}
\]

\[
\equiv \sim (p \land \sim q) \quad \text{By De Morgan’s Law}
\]

\[
\equiv \sim ((p \downarrow p) \downarrow (\sim q \downarrow \sim q)) \quad \text{By Part (b)}
\]

\[
\equiv \sim ((p \downarrow p) \downarrow ((q \downarrow q) \downarrow q \downarrow q)) \quad \text{By Part (a)}
\]

\[
\equiv (((p \downarrow p) \downarrow ((q \downarrow q) \downarrow q \downarrow q)) \downarrow ((p \downarrow p) \downarrow ((q \downarrow q) \downarrow q \downarrow q))) \quad \text{By Part (a)}
\]

**Question 3.** Write a negation for each of the following statement.

(a) $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, \exists z \in \mathbb{R}$ such that $x = y + z$. (1 marks)

(b) $\exists$ a book $b$ such that $\forall$ people $p$, $p$ has read $b$. (1 marks)

**Solution:**

(a) $\exists x \in \mathbb{R}, \exists y \in \mathbb{R}$ such that $\forall z \in \mathbb{R}$ $x \neq y + z$.

(b) $\forall$ book $b \exists$ a people $p$ such that $p$ has not read $b$.

**Question 4.** Prove or disprove: For all real numbers $x$, $\lfloor x^2 \rfloor = (\lfloor x \rfloor)^2$. (2 marks)

**Solution:** The statement is false. As a counter-example let $x = \frac{3}{2}$. Then

\[
\lfloor (\frac{3}{2})^2 \rfloor = \lfloor \frac{9}{4} \rfloor = 2.
\]

And

\[
(\lfloor \frac{3}{2} \rfloor)^2 = 1^2 = 1.
\]

So $\lfloor x^2 \rfloor \neq (\lfloor x \rfloor)^2$ in general.

**Question 5** Solve the following Linear Diophantine Equation. (4 marks)

\[
330x + 156y = -6.
\]
**Solution:** First using Euclidean Algorithm we find \( \gcd(330, 156) \).

\[
\begin{align*}
330 & = 2 \times 156 + 18 \\
156 & = 8 \times 18 + 12 \\
18 & = 1 \times 12 + 6 \\
6 & = 2 \times 6 + 0
\end{align*}
\]

So \( \gcd(330, 156) = 6 \). Now since \( 6 \mid -6 \) the linear equation has a solution in integers for \( x \) and \( y \). To find this solution we proceed as follows:

\[
\begin{align*}
6 & = 18 - 1 \times 12 \\
& = 18 - 1 \times (156 - 8 \times 18) \\
& = -156 + 9 \times 18 \\
& = -156 + 9 \times (330 - 2 \times 156) \\
& = 9 \times 330 - 19 \times 156
\end{align*}
\]

Now from \( 6 = 9 \times 330 - 19 \times 156 \) we obtain \( -6 = -9 \times 330 + 19 \times 156 \). So \( x = -9 \) and \( y = 19 \).