1. Which of the following are linear equations in \( x_1, x_2 \) and \( x_3 \)?

(a) \( x_1 + 2x_2 + 7x_3 = 6 \)  
(b) \( x_1x_3 + x_2 = 3 \)

(c) \( x_1 + 3x_3 = -2x_2 + \frac{1}{5} \)  
(d) \( x_1 = 3\sqrt{x_3} + x_2^2 \)

(e) \( x_1 = x_2 \)  
(f) \( x_1^2 + x_2^2 + 2x_3^2 = 3^2 \)

2. Classify the following systems of equations as non-linear or linear. Further classify the linear systems as non-homogenous or homogenous and form the augmented matrix of the system of linear equations.

(a) \( \begin{align*} 
0 & -2x_2 &= 0 \\
3x_1 + 4x_2 &= -1 \\
2x_1 - x_2 &= 3 
\end{align*} \)  
(b) \( \begin{align*} 
0 & -3x_2 + x_3 &= 0 \\
5x_1 - 2x_2 - 3x_3 &= 0 \\
-7x_1 + x_2 + 2x_3 &= 0 
\end{align*} \) 

(c) \( \begin{align*} 
2x_1 - 3x_2^2 + x_3 &= 0 \\
-5x_1 - 5x_2^2 - x_3 &= 0 \\
3x_1 + x_2^2 + x_3 &= 0 
\end{align*} \)  
(d) \( \begin{align*} 
x_1 + x_3 &= 1 \\
-x_1 + 2x_2 - x_3 &= 3 
\end{align*} \)

3. Using elementary row operations convert the following matrices to reduced row-echelon form

(a) \( \begin{bmatrix} 0 & 0 & 0 \\
0 & 2 & 4 \end{bmatrix} \)  
(b) \( \begin{bmatrix} 0 & 1 & 3 \\
1 & 2 & 4 \end{bmatrix} \)  
(c) \( \begin{bmatrix} 1 & 1 & 0 \\
1 & 1 & 1 \\
1 & 0 & 0 \end{bmatrix} \)  
(d) \( \begin{bmatrix} 2 & 1 & -1 \\
0 & 3 & 1 \\
-4 & 0 & 0 \end{bmatrix} \)

4. Solve the following homogenous system by finding the reduced row-echelon form of the coefficient matrix:

\[3x_1 + x_2 + x_3 + x_4 = 0 \]
\[5x_1 - x_2 + x_3 - x_4 = 0 \]

5. Solve the following homogenous system by finding the reduced row-echelon form
of the coefficient matrix:

\[-3x_1 + x_2 + x_3 + x_4 = 0\]
\[x_1 - 3x_2 + x_3 + x_4 = 0\]
\[x_1 + x_2 - 3x_3 + x_4 = 0\]
\[x_1 + x_2 + x_3 - 3x_4 = 0\]

6. Consider the system of equations

\[x + y + 2z = a\]
\[x + z = b\]
\[2x + y + 3z = c\]

Show that in order for this system to be consistent, \(a, b\) and \(c\) must satisfy \(c = a + b\).

7. For which values of \(a\) will the following system have no solutions? Exactly one solution? Infinitely many solutions?

\[x + 2y + 3z = 4\]
\[3x - y + 5z = 2\]
\[4x + y + (a^2 - 14)z = a + 2\]

8. Let \(A\) be \(n \times n\), then

(a) If \(A^2 = 0\) prove that \(A\) is singular

(b) If \(A^2 = A\) and \(A \neq I_n\) prove that \(A\) is singular.

9. Calculate \(A^0, A^T\) and \(A^{-1}\) when

\[
A = \begin{bmatrix}
2 & 0 & 0 \\
0 & -3 & 0 \\
0 & 0 & 4
\end{bmatrix}
\]
1. Let $V$ be a real vector space, $p$ a fixed vector in $V$. Let us define a new addition and scalar multiplication on $V$ by the formulae

\[ u \oplus v = u + v + p \]
\[ t \otimes v = tv + (t - 1)p \]

for all $u, v \in V$, $t \in \mathbb{R}$. Show that $V$ is also a vector space under the new operations, with $-p$ as the zero vector and $-(v + 2p)$ as the additive inverse of $v$.

2. Define a new addition on $\mathbb{R}^2$ by

\[ (x_1, y_1) \oplus (x_2, y_2) = ([x_1^3 + x_2^3]^\frac{1}{3}, [y_1^3 + y_2^3]^\frac{1}{3}) \]

Show that with this new addition and the usual scalar multiplication, $(0, 0)$ is still the additive identity and $(-x, -y)$ the additive inverse of $(x, y)$. Also verify that all the vector space axioms hold, apart from the axiom

\[ (s + t)u = su + tu \]

3. If $U_1$ and $U_2$ are subspaces of a vector space $V$ and $U_1 \cup U_2$ is also a subspace of $V$, prove that $U_1 \subset U_2$ or $U_2 \subset U_1$. (Hint: Prove that $U_1 \not\subset U_2$ and $U_2 \not\subset U_1$ implies $U_1 \cup U_2$ is not closed under addition.)

4. If $U$ and $V$ are subspaces of a vector space $W$, prove that the subset of $W$ defined by $U + V = \{u + v \mid u \in U, v \in V\}$ is also a subspace of $W$.

5. Let $u_1, u_2, \ldots, u_m$ and $v_1, v_2, \ldots, v_m$ belong to $V$. Let

\[ U_1 = \text{span}(u_1, u_2, \ldots, u_m) \]
\[ U_2 = \text{span}(v_1, v_2, \ldots, v_m) \]

prove that

\[ U_1 + U_2 = \text{span}(u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_m) \]
6. If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ belong to the real vector space $V$, prove that

$$\text{span}(\mathbf{u} + \mathbf{v}, \mathbf{v} + \mathbf{w}, \mathbf{w} + \mathbf{u}) = \text{span}(\mathbf{u}, \mathbf{v}, \mathbf{w})$$

7. $U$ and $V$ are subspaces of $\mathbb{R}^3$ defined by

$$U = \{(x, y, z) | x + y + z = 0\} \quad \text{and} \quad V = \{(x, y, z) | x - y - z = 0\}$$

Find spanning families for $U$ and $V$ and prove that $U + V = \mathbb{R}^3$.

8. Which of the following subsets of $\mathbb{R}^2$ are subspaces of $\mathbb{R}^2$?

(a) $\{(x, y) | x = 3y\}$

(b) $\{(x, y) | x^2 = y^2\}$

(c) $\{(x, y) | x + y = 1\}$

(d) $\{(x, y) | x \geq 0 \text{ and } y \geq 0\}$

9. Let $A \in M_{n \times n}(\mathbb{R})$ and let $U$ be the subset of $M_{n \times n}(\mathbb{R})$ defined by

$$U = \{X \in M_{n \times n}(\mathbb{R}) | AX = XA\}$$

(a) Prove that $U$ is a subspace of $M_{n \times n}(\mathbb{R})$.

(b) Let $V$ be the set of matrices of the form

$$a_0 I_n + a_1 A + \cdots + a_m A^m, \quad a_0, a_1, \ldots, a_m \in \mathbb{R}.$$

Prove that $V$ is a subspace of $M_{n \times n}(\mathbb{R})$ and that $V \subset U$.

(c) Find spanning families for $U$ and $V$ when

- $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

- $A = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$, $\lambda \neq \mu$

(Hint: If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $A^2 = (a + d)A - (ad - bc)I_2$.)
10. Which of the following sets of vectors in $\mathbb{R}^3$ are linearly independent?
   (a) $(2,-1,2),(3,0,1),(2,2,2)$
   (b) $(3,1,1),(2,-1,5),(1,7,-17)$
   (c) $(6,0,-1),(1,1,4)$
   (d) $(1,3,3),(0,1,4),(5,6,3),(7,2,-1)$

11. Which of the following sets of vectors in $P_2$ are linearly independent?
   (a) $2 - x + 4x^2$, $3 + 6x + 2x^2$, $2 + 10x - 4x^2$
   (b) $3 + x + x^2$, $2 - x + 5x^2$, $4 - 3x^2$
   (c) $6 - x^2$, $1 + x + 4x^2$
   (d) $1 + 3x + 3x^2$, $x + 4x^2$, $5 + 6x + 3x^2$, $7 + 2x - x^2$

12. Let $\alpha$ and $\beta$ be distinct real numbers. Prove that the vectors $(1, \alpha)$ and $(1, \beta)$ are linearly independent.

13. Let $\alpha$, $\beta$ and $\gamma$ be distinct real numbers. Prove that the vectors $(1, \alpha, \alpha^2)$, $(1, \beta, \beta^2)$ and $(1, \gamma, \gamma^2)$ are linearly independent.

14. Let $u_1, u_2, \ldots, u_n$ be a linearly independent family of vectors in $V$ and let vectors $v_1, v_2, \ldots, v_m \in V$ be defined by
   
   $$ v_i = \sum_{j=1}^{n} a_{ij} u_j, \quad 1 \leq i \leq m $$

   Prove that $v_1, v_2, \ldots, v_m$ are linearly independent if and only if the rows of the matrix $A = [a_{ij}]$ are linearly independent.

15. Prove that three vectors in $\mathbb{R}^n$ are linearly dependent if and only if they lie in a plane.
Tutorial Sheet Three

1. Explain why the following sets of vectors are not bases for the indicated vector spaces. (Solve this problem by inspection).

   (a) $\mathbf{u}_1 = (1, 2), \mathbf{u}_2 = (0, 3), \mathbf{u}_3 = (2, 7)$ for $\mathbb{R}^2$
   (b) $\mathbf{u}_1 = (1, 2, 1), \mathbf{u}_2 = (0, 3, 2)$ for $\mathbb{R}^3$
   (c) $\mathbf{p}_1 = 1 + x + x^2, \mathbf{p}_2 = x - 1$ for $P_2$

2. Which of the following sets of vectors are bases for $\mathbb{R}^3$?

   (a) $(1,0,0),(2,2,0),(3,3,3)$
   (b) $(3,1,4),(2,5,6),(1,4,8)$
   (c) $(2,3,1),(4,1,1),(0,-7,-1)$
   (d) $(1,6,4),(2,4,-1),(-1,2,5)$

3. Which of the following sets of vectors are bases for $P_2$?

   (a) $1 - 3x + 2x^2, 1 + x + 4x^2, 1 - 7x$
   (b) $4 + 6x + x^2, -1 + 4x + 2x^2, 5 + 2x - x^2$
   (c) $1 + x + x^2, x + x^2, x^2$
   (d) $-4 + x + 3x^2, 6 + 5x + 2x^2, 8 + 4x + x^2$

4. Show that the following set of vectors is a basis for $M_{2\times2}(\mathbb{R})$.

   \[
   \begin{bmatrix}
   3 & 6 \\
   3 & -6
   \end{bmatrix}
   ,
   \begin{bmatrix}
   0 & -1 \\
   -1 & 0
   \end{bmatrix}
   ,
   \begin{bmatrix}
   0 & -8 \\
   -12 & -4
   \end{bmatrix}
   ,
   \begin{bmatrix}
   1 & 0
   \end{bmatrix}
   \]

   In Questions 5 and 6 determine the dimension and a basis for the solution space of the system

5. $3x_1 + x_2 + 2x_3 = 0$
   $4x_1 + 5x_3 = 0$

6. $3x_1 + x_2 + x_3 + x_4 = 0$
   $5x_1 - x_2 + x_3 - x_4 = 0$
7. Determine bases for the following subspaces of $\mathbb{R}^3$.
(a) The plane $3x - 2y + 5z = 0$
(b) The plane $x - y = 0$
(c) The line described by the parametric equations
\[
\begin{align*}
x &= t \\
y &= -t \\
z &= 4t
\end{align*}
\]

8. Determine the dimensions of the following subspaces of $\mathbb{R}^4$.
(a) All vectors of the form $(a, b, c, 0)$
(b) All vectors of the form $(a, b, c, d)$ where $d = a + b$ and $c = a - b$
(c) All vectors of the form $(a, b, c, d)$ where $a = b = c = d$

9. Determine the dimension of the subspace of $P^3$ consisting of all polynomials $a_0 + a_1x + a_2x^2 + a_3x^3$ for which $a_0 = 0$.

10. Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a basis for a vector space $V$. Show that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is also a basis, where $\mathbf{u}_1 = \mathbf{v}_1$, $\mathbf{u}_2 = \mathbf{v}_1 + \mathbf{v}_2$ and $\mathbf{u}_3 = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3$.

11. Show that the vector space of all real-valued functions defined on the entire real line is infinite dimensional. (Hint: Assume it is finite dimensional with dimension $n$, and obtain a contradiction by producing $n+1$ linearly independent vectors.)

12. Let $V$ be a subspace of a finite dimensional vector space $W$. Show that $\dim(V) \leq \dim(W)$.

13. Show that the only subspaces of $\mathbb{R}^3$ are lines through the origin, planes through the origin, the zero subspace, and $\mathbb{R}^3$ itself. (Hint: It is known that the subspaces of $\mathbb{R}^3$ must be 0-dimensional, 1-dimensional, 2-dimensional or 3-dimensional.)
14. Find the coordinate vector of \( \mathbf{v} = (7, 4)^T \) relative to the basis \((3, 2)^T, (1, 1)^T \) of \( \mathbb{R}^2 \).

15. \( \mathbf{v}_1 = (1, 1, 1)^T, \mathbf{v}_2 = (2, 3, 2)^T, \mathbf{v}_3 = (1, 5, 4)^T \) form a basis \( \beta \) for \( \mathbb{R}^3 \). Vectors \( \mathbf{u}_1 = (1, 1, 0)^T, \mathbf{u}_2 = (1, 2, 0)^T, \mathbf{u}_3 = (1, 2, 1)^T \) form a basis \( \gamma \) for \( \mathbb{R}^3 \). Find the change of basis matrix \( [P]_\beta^\gamma \). Use this matrix to find \( [3\mathbf{v}_1 + 2\mathbf{v}_2 - \mathbf{v}_3]_\gamma \).

16. Let \( V \) be the vector space \( \mathbb{R}^2 \). Let \( \beta = \{1, x, x^2\} \) and \( \gamma = \{1 + x, 1 + x^2, x + x^2\} \) be two bases for \( V \). Calculate the change of basis matrix \( [P]_\beta^\gamma \). Use this to calculate \([3 + x + -2x^2]_\gamma \).

17. Let \( \beta \) be a basis for an \( n \)-dimensional vector space \( V \). Show that if \( \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r \) form a linearly independent set of vectors in \( V \), then the coordinate vectors \([\mathbf{v}_1]_\beta, [\mathbf{v}_2]_\beta, \ldots, [\mathbf{v}_r]_\beta \) form a linearly independent set in \( \mathbb{R}^n \). Similarly show that if \( \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r \) span \( V \), then the coordinate vectors \([\mathbf{v}_1]_\beta, [\mathbf{v}_2]_\beta, \ldots, [\mathbf{v}_r]_\beta \) span \( \mathbb{R}^n \). Conclude that \( \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \) forms a basis for \( V \) if and only if \([\mathbf{v}_1]_\beta, [\mathbf{v}_2]_\beta, \ldots, [\mathbf{v}_n]_\beta \) form a basis for \( \mathbb{R}^n \).

18. Find an orthonormal basis for the subspace of \( \mathbb{R}^4 \) spanned by

\[
\mathbf{u}_1 = (1, 1, 1, 1)^T, \quad \mathbf{u}_2 = (0, 1, 1, 1)^T, \quad \mathbf{u}_3 = (0, 0, 1, 1)^T
\]

Extend this to an orthonormal basis for \( \mathbb{R}^4 \).
1. Find bases for the row space, the column space and the null space of the following matrices. Verify for each matrix that \( \dim R(A) = \dim C(A) \) and that \( \text{rank}(A) + \text{nullity}(A) = n \).

\[
\begin{bmatrix}
1 & -3 \\
2 & -6
\end{bmatrix}
\begin{bmatrix}
1 & 2 & 1 \\
2 & 4 & 6 \\
0 & 0 & -8
\end{bmatrix}
\begin{bmatrix}
-3 & 5 & 1 & 2 \\
7 & 2 & 0 & -4 \\
-8 & 3 & 1 & 6
\end{bmatrix}
\begin{bmatrix}
1 & -3 & 2 & 2 & 1 \\
0 & 3 & 6 & 0 & -2 \\
2 & -3 & -2 & 4 & 4 \\
3 & -3 & 6 & 6 & 3 \\
5 & -3 & 10 & 10 & 5
\end{bmatrix}
\]

2. Find a basis for the subspace of \( \mathbb{R}^4 \) spanned by the given vectors.
   (a) \((1,1,-4,-3),(2,0,2,-2),(2,-1,3,2)\)
   (b) \((-1,1,-2,0),(3,3,6,0),(9,0,0,3)\)
   (c) \((1,1,0,0),(0,0,1,1),(-2,0,2,2),(0,-3,0,3)\)

3. Find a basis for the subspace of \( P_2 \) spanned by the given vectors.
   (a) \(-1 + x - 2x^2, 3 + 3x + 6x^2, 9\)
   (b) \(1 + x, x^2, -2 + 2x^2, -3x\)
   (c) \(1 + x - 3x^2, 2 + 2x - 6x^2, 3 + 3x - 9x^2\)

4. Find a basis for the subspace of \( M_{2 \times 2}(\mathbb{R}) \) spanned by the given vectors.
   (a) \[
\begin{bmatrix}
-1 & -1 \\
4 & 3
\end{bmatrix}
\begin{bmatrix}
2 & 0 \\
2 & -2
\end{bmatrix}
\begin{bmatrix}
2 & -1 \\
3 & 2
\end{bmatrix}
\]
   (b) \[
\begin{bmatrix}
-1 & 2 \\
-2 & 0
\end{bmatrix}
\begin{bmatrix}
3 & -3 \\
6 & 0
\end{bmatrix}
\begin{bmatrix}
9 & 0 \\
0 & 3
\end{bmatrix}
\]

5. Show that the nonzero row vectors in any row-echelon form of a matrix \( A \) form a basis for the row space of \( A \).

6. Show that row vectors in an \( n \times n \) invertible matrix \( A \) form a basis for \( \mathbb{R}^n \).
7. $U$ and $V$ are subspaces of $\mathbb{R}^5$ where, $U = \text{span}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$, $V = \text{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$, where $u_1, u_2, u_3, v_1, v_2, v_3$ are the respective columns of the matrix $A$:

$$
A = \begin{bmatrix}
1 & 1 & 2 & 1 & 2 & 1 \\
3 & 4 & 9 & 6 & 8 & 3 \\
-3 & -1 & 0 & 2 & -1 & -1 \\
-1 & -2 & -5 & -2 & -6 & -5 \\
-4 & -2 & -2 & 3 & -5 & -6
\end{bmatrix}
$$

(a) Assuming that $A$ has reduced row-echelon form

$$
B = \begin{bmatrix}
1 & 0 & -1 & 0 & -2 & -3 \\
0 & 1 & 3 & 0 & 5 & 6 \\
0 & 0 & 0 & 1 & -1 & -2 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

find bases for each of the subspaces $U, V, U + V$.

8. (a) If $a_1, a_2, \ldots, a_n$ are not all zero, prove that the set

$$
U = \{(x_1, x_2, \ldots, x_n)^T \mid a_1x_1 + \cdots + a_nx_n = 0\}
$$

is a subspace of $\mathbb{R}^n$ with dimension equal to $n - 1$.

(b) Prove conversely that any subspace $U$ of $\mathbb{R}^n$ having dimension equal to $n - 1$ must have the form (1).

9. $A$ is a $3 \times 3$ matrix such that $A^2 = 0$ and $A \neq 0$. Prove that

(a) $C(A) \subseteq N(A)$. (Hint: Let $X \in C(A)$. Then $X = AY$ for some $Y \in \mathbb{R}^3$.)

(b) $\text{rank}(A) = 1$. (Hint: Use the $\text{rank} + \text{nullity}$ theorem.)

(c) Exhibit a nonzero $3 \times 3$ matrix $A$ such that $A^2 = 0$.

10. Suppose that $A \in M_{n \times n}(\mathbb{R})$ satisfies $A^2 = A$. Prove the following
(a) If $Y \in C(A)$, then $AY = Y$.

(b) $N(A)\tilde{C}(A) = \{0\}$.

(c) $\mathbb{R}^n = N(A) + C(A)$. 
Tutorial Sheet Five

1. \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) is a linear transformation which maps \((1, 2)^T\) to \((-2, 3)^T\) and \((1, -1)^T\) to \((5, 2)^T\). Find \( T(\mathbf{v}) \) when \( \mathbf{v} = (7, 5)^T \).

2. Let \( T : U \to V \) be a linear transformation. If \( \text{Ker}T = \{0\} \) and \( \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n \) are linearly independent in \( U \), prove that \( T(\mathbf{u}_1), \ldots, T(\mathbf{u}_2) \) are linearly independent in \( V \).

3. Let \( T : U \to V \) be a linear transformation. Prove that \( T \) is 1-1 (ie. \( T(\mathbf{u}) = T(\mathbf{v}) \) implies \( \mathbf{u} = \mathbf{v} \)) if and only if \( \text{Ker}T = \{0\} \).

4. Let \( T : U \to V \) be a linear transformation. Using the \textit{rank + nullity} theorem, prove that
   
   (a) \( \text{rank}(T) \leq \dim(U) \)
   
   (b) \( \text{Ker}(T) = \{0\} \Rightarrow \dim(U) \leq \dim(V) \)

5. Let \( U \) be a vector space with basis \( \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \). \( T : U \to U \) is the linear transformation defined by

\[
\begin{align*}
T(\mathbf{u}_1) &= \mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3 \\
T(\mathbf{u}_2) &= \mathbf{u}_1 - \mathbf{u}_2 + \mathbf{u}_3 \\
T(\mathbf{u}_3) &= 2\mathbf{u}_1 + 2\mathbf{u}_3
\end{align*}
\]

Find bases for \( \text{Ker}(T), \text{Im}(T) \). Also find \( \text{rank}(T) \) and \( \text{nullity}(T) \).

6. Let \( U \) be a vector space with basis \( \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \). \( T : U \to U \) is the linear transformation defined by

\[
\begin{align*}
T(\mathbf{u}_1) &= \mathbf{u}_3 & T(\mathbf{u}_2) &= -\mathbf{u}_3 & T(\mathbf{u}_3) &= \mathbf{u}_1 + \mathbf{u}_2
\end{align*}
\]

Find bases for \( \text{Ker}(T), \text{Im}(T) \). Also find \( \text{rank}(T) \) and \( \text{nullity}(T) \).
7. Let \( A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \) and let \( T : M_{2 \times 2}(\mathbb{R}) \to M_{2 \times 2}(\mathbb{R}) \) be the linear transformation defined by \( T(X) = AX - XA \). Prove that
\[
\text{Im}(T) = \text{span}\left\{ \begin{bmatrix} -1 & -2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 2 & -2 \end{bmatrix} \right\}, \quad \text{Ker}T = \text{span}\{I_2, A\}
\]
Write down \( \text{rank}(T) \) and \( \text{nullity}(T) \).

8. If \( T : U \to V \) is a linear transformation, prove that \( \text{Im}(T) \) is a subspace of \( V \).

9. Let \( T : U \to V \) be a linear transformation, where \( \dim(U) = \dim(V) \). If \( \text{Ker}(T) = \{0\} \), deduce that \( \text{Im}(T) = V \).

10. Let \( \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \) be a basis for \( \mathbb{R}^3 \) and \( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \) be any vectors in \( \mathbb{R}^3 \). If \( T \) is a linear transformation such that \( T(\mathbf{u}_i) = \mathbf{v}_i \) for \( i = 1, 2, 3 \), show that \( T = T_A \), where \( A \) is the \( 3 \times 3 \) matrix
\[
A = \begin{bmatrix} \mathbf{v}_1 & | & \mathbf{v}_2 & | & \mathbf{v}_3 \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 & | & \mathbf{u}_2 & | & \mathbf{u}_3 \end{bmatrix}^{-1}
\]

11. Suppose \( V = M_{2 \times 2}(\mathbb{R}) \) and \( \beta : E_{11}, E_{12}, E_{21}, E_{22} \) is the standard basis for \( V \). Mappings \( S, T : V \to V \) are defined by
\[
T(A) = \frac{1}{2}(A - A^T), \quad S(A) = \frac{1}{2}(A + A^T)
\]
(a) Prove that \( S \) and \( T \) are linear.
(b) Find \( [S]_{\beta}^\beta \) and \( [T]_{\beta}^\beta \).
(c) Find bases for \( \text{Ker}(S) \) and \( \text{Im}(S) \), \( \text{Ker}(T) \) and \( \text{Im}(T) \).
(d) Prove that \( S^2 = S, T^2 = T \).
(e) Prove that \( ST = 0, TS = 0 \)
(f) Prove that \( S + T = I_V \), where \( S + T \) is the linear mapping defined by \( (S + T)(\mathbf{v}) = S(\mathbf{v}) + T(\mathbf{v}) \).
12. Let \( \gamma : \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \) be the standard basis of unit vectors for \( V = \mathbb{R}^3 \) and let \( \beta : \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \) be the basis of \( \mathbb{R}^3 \) given by

\[
\mathbf{v}_1 = [1, 1, -1]^T, \quad \mathbf{v}_2 = [2, 1, 3]^T, \quad \mathbf{v}_3 = [0, 1, 1]^T
\]

Find \( [I_V]_\beta \) and \( [I_V]_\gamma \).

13. Let \( A \) and \( B \) be non-singular \( n \times n \) matrices over \( \mathbb{R} \) and let \( V = M_{n \times n}(\mathbb{R}) \). Let \( X \in V \). Show that the mapping \( T : V \to V \) defined by \( T(X) = AXB \) has the property that \( \text{Ker}(T) = \{0\} \) and \( \text{Im}(T) = V \).

14. A mapping \( T : P_2[\mathbb{R}] \to \mathbb{R}^3 \) is defined by

\[
T(f(x)) = \begin{bmatrix} f(1) \\ f(0) \\ f(-1) \end{bmatrix}
\]

(a) Prove that \( T \) is a linear transformation.

(b) If \( S : \mathbb{R}^3 \to P_2[\mathbb{R}] \) is the linear transformation defined by

\[
S \left( \begin{bmatrix} a \\ b \\ c \end{bmatrix} \right) = b + \frac{a - c}{2}x + \frac{a - 2b + c}{2}x^2
\]

verify that \( ST = I_{P_2[\mathbb{R}]} \) and \( TS = I_{\mathbb{R}^3} \).

15. Let \( T : P_2[\mathbb{R}] \to P_2[\mathbb{R}] \) be given by \( T(f(x)) = f'(x)g(x) + 2f(x) \), where \( g(x) = 3 + x \) and \( f'(x) \) is the formal derivative of \( f \) (ie. if \( f = a_0 + a_1x + a_2x^2 \), then \( f'(x) = a_1 + 2a_2x \), where \( a_0, a_1, a_2 \in \mathbb{R} \)).

Also let \( S : P_2[\mathbb{R}] \to \mathbb{R}^3 \) be defined by \( S(a + bx + cx^2) = [a, b, c, a - b]^T \), where \( a, b, c \in \mathbb{R} \).

Let \( \beta : 1, x, x^2 \) and \( \gamma : \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \) be the standard bases for \( P_2[\mathbb{R}] \) and \( \mathbb{R}^3 \) respectively.

Find \( [S]_\beta^\gamma, [T]_\beta^\gamma \) and \( [ST]_\beta^\gamma \) and verify that \( [ST]_\beta^\gamma = [S]_\beta^\gamma[T]_\beta^\gamma \).

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16. Let $T : P_4[\mathbb{R}] \rightarrow P_4[\mathbb{R}]$ be the linear transformation defined by
\[
T(f(x)) = \frac{1}{2}(f(x) + f(-x)).
\]
(a) Prove that $T^2 = T$.

(b) For the basis $\beta : 1, x^2, x^4, x, x^3$ of $P_4[\mathbb{R}]$, find $[T]_\beta^\beta$.

In exercises 17 and 18 find $[T]_\beta^\beta$ and use Theorem 6.12 to calculate $[T]_\gamma^\gamma$.

17. Let $\beta = \{u_1, u_2\}$ and $\gamma = \{v_1, v_2\}$ be two bases for $\mathbb{R}^2$, where
\[
u_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \nu_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{and} \quad v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad v_2 = \begin{bmatrix} -3 \\ 4 \end{bmatrix}
\]

$T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is defined by
\[
T \left( \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} x_1 - 2x_2 \\ -x_2 \end{bmatrix}
\]

18. Let $\beta = \{u_1, u_2\}$ and $\gamma = \{v_1, v_2\}$ be two bases for $\mathbb{R}^2$, where
\[
u_1 = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \quad \nu_2 = \begin{bmatrix} 4 \\ -1 \end{bmatrix} \quad \text{and} \quad v_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad v_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}
\]

$T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is defined by
\[
T \left( \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} x_1 + 7x_2 \\ 3x_1 - 4x_2 \end{bmatrix}
\]

19. Prove that similar matrices, have the same rank and nullity.

20. Prove that if $A$ and $B$ are similar matrices, then $A^k$ and $B^k$ are also similar, where $k$ is any positive integer.
1. If \( \lambda \) is an eigenvalue of \( A \), prove that \( \lambda^2 \) is an eigenvalue of \( A^2 \).

2. Let \( A = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \)

   (a) Verify that \( ch_A(x) = (x - 1)(x + \frac{1}{2})^2 \)

   (b) Find \( g_A(1) \) and \( g_A(-\frac{1}{2}) \) and deduce that \( A \) is diagonalizable.

   (c) Find a non-singular matrix \( P \in M_{n \times n}(\mathbb{R}) \) such that \( P^{-1}AP \) is diagonal.

3. For each of the following matrices, determine if they are diagonalizable. If so, find a matrix \( P \) that diagonalizes \( A \), and determine \( P^{-1}AP \).

\[
\begin{bmatrix}
5 & 0 & 0 \\
1 & 5 & 0 \\
0 & 1 & 5
\end{bmatrix}
\begin{bmatrix}
2 & 3 & -6 \\
3 & 5 & 8 \\
-6 & 8 & 4
\end{bmatrix}
\begin{bmatrix}
-1 & 4 & -2 \\
-3 & 4 & 0 \\
-3 & 1 & 3
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & 2 & 0 \\
3 & 2 & 3
\end{bmatrix}
\]

4. Evaluate \( A^m \) for each matrix \( A \). Use diagonalization to shorten the process.

   (a) \( \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} \)

   (b) \( \begin{bmatrix} 2 & 0 & -2 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \)

5. Solve the following system of linear differential equations by using an appropriate change of variables.

\[
x'_1 = -x_1 + 2x_2 + 3x_3 \\
x'_2 = 2x_1 - 3x_2 + 4x_3 \\
x'_3 = 3x_1 + 4x_2 - 2x_3
\]
6. Let

\[ A = \begin{bmatrix}
1 & -4 & 0 \\
-4 & 3 & -4 \\
0 & -4 & 5
\end{bmatrix} \]

Find an orthogonal matrix \( P \) such that \( P^T AP = D \) where \( D \) is diagonal.

7. Let

\[ A = \begin{bmatrix}
5 & 2 & -2 \\
2 & 5 & -2 \\
-2 & -2 & 5
\end{bmatrix} \]

Find an orthogonal matrix \( P \) such that \( P^T AP = D \) where \( D \) is diagonal.

8. If \( A \) is a real symmetric matrix and \( \lambda \) and \( \mu \) are distinct eigenvalues of \( A \), with corresponding eigenvectors \( X \) and \( Y \), prove that \( X^T Y = 0 \).

9. If \( A \) is a real symmetric matrix, prove that \( N(A^2) = N(A) \). (Hint: Assume \( A^2 X = 0 \). Then \( A^T AX = 0 \) etc.)