2. Partial Derivative and Tangent Planes

2.4. Gradients and Directional Derivatives

We can use partial derivatives to work out the slope of \( z = f(x, y) \) in the \( x \) direction; \( \frac{\partial f}{\partial x} \), or in the \( y \) direction; \( \frac{\partial f}{\partial y} \). But what is the slope in the \( y = x \) direction?

First we need a more precise definition of the direction using vectors:

Let \( \mathbf{i} \) be a unit vector in the \( x \) direction and let \( \mathbf{j} \) be a unit vector in the \( y \) direction. A unit vector is a vector whose magnitude is 1.

A vector in the \( y = x \) direction is \( \mathbf{v} = \mathbf{i} + \mathbf{j} \). But this is not a unit vector, as it's magnitude \( ||\mathbf{v}|| = \sqrt{1^2 + 1^2} = \sqrt{2} \). So we divide by \( ||\mathbf{v}|| \) to get a unit vector.

\[
\mathbf{u} = \frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j}) \text{ is a unit vector in the } y = x \text{ direction.}
\]

Suppose as an example we wanted to find the slope of \( f(x, y) = 4 - x^2 - 4y^2 \) at \((a, b)\) in the \( y = x \) direction. Say we move a short distance \( h \) in that direction. Then we could get an approximate slope by calculating the corresponding change in height \( f \), that is \( \Delta f \). Then the slope is \( \frac{\Delta f}{h} \).

Now since \( \mathbf{u} = \frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j}) \) is a unit vector in the \( y = x \) direction it follows that

\[
h\mathbf{u} = \frac{h}{\sqrt{2}}\mathbf{i} + \frac{h}{\sqrt{2}}\mathbf{j}
\]

is a vector of length \( h \) in that direction.

So moving a distance \( h \) in the \( y = x \) direction implies a distance \( \Delta x = \frac{h}{\sqrt{2}} \) in the \( x \) direction and a distance \( \Delta y = \frac{h}{\sqrt{2}} \) in the \( y \) direction.

Now we can see that the corresponding change in height would be

\[
\Delta f = f(a + \frac{h}{\sqrt{2}}, b + \frac{h}{2}) - f(a, b).
\]

But \( h \) is small so we can use linear approximations;

\[
\Delta f \simeq \frac{\partial f}{\partial x} \Delta x + \frac{\partial f}{\partial y} \Delta y
\]
where here \( \Delta x = \frac{h}{\sqrt{2}} \) and \( \Delta y = \frac{h}{\sqrt{2}} \) and we know how to calculate

\[
\frac{\partial f}{\partial x}(a, b) \quad \text{and} \quad \frac{\partial f}{\partial y}(a, b).
\]

So

\[
\Delta f \approx \frac{\partial f}{\partial x} \frac{h}{\sqrt{2}} + \frac{\partial f}{\partial y} \frac{h}{\sqrt{2}}.
\]

which implies that the slope in the \( \left( \frac{1}{\sqrt{2}} i + \frac{1}{\sqrt{2}} j \right) \) direction is

\[
\frac{\Delta f}{h} \approx \frac{1}{\sqrt{2}} \frac{\partial f}{\partial x} + \frac{1}{\sqrt{2}} \frac{\partial f}{\partial y}.
\]

Now let \( h \to 0 \). Then the rate of change of \( f \) is \( \frac{1}{\sqrt{2}} \frac{\partial f}{\partial x} + \frac{1}{\sqrt{2}} \frac{\partial f}{\partial y} \) in the direction \( u = \frac{1}{\sqrt{2}} i + \frac{1}{\sqrt{2}} j \).

**Back to the example.**

If \( f(x, y) = 4 - x^2 - 4y^2 \) and \( (a, b) = (1, 1) \) then

\[
\frac{\partial f}{\partial x}(1, 1) = -2x \bigg|_{(1,1)} = -2
\]

and

\[
\frac{\partial f}{\partial y}(1, 1) = -8y \bigg|_{(1,1)} = -8,
\]

So the slope in the \( u = \frac{1}{\sqrt{2}} i + \frac{1}{\sqrt{2}} j \) direction is

\[
= \frac{1}{\sqrt{2}}(-2) + \frac{1}{\sqrt{2}}(-8) = \frac{-10}{\sqrt{2}} = -5\sqrt{2}.
\]

The rate of change of \( f \) in the direction \( u \) is called a **directional derivative**.

**Directional Derivatives.**

The directional derivative in the direction \( u = u_1 i + u_2 j \), where \( ||u|| = 1 \), at \( (a, b) \) is

\[
f_u (a, b) = \frac{\partial f}{\partial x}(a, b)u_1 + \frac{\partial f}{\partial y}(a, b)u_2
\]

and it is simply the slope of the surface \( f(x, y) \) in the direction \( u \). (But remember \( u \) must be a unit vector.)
**Example.** If \( f(x, y) = x^2 - 3y^2 + 6y \). Find the slope at \((1, 0)\) in the direction \( i - 4j \).

Now \( \|i - 4j\| = \sqrt{1 + 16} = \sqrt{17} \).

So \( u = \frac{1}{\sqrt{17}}(i - 4j) \) is a unit vector in the direction \( i - 4j \).

Now slope \( \frac{\partial f}{\partial x}(1, 0) = \frac{\partial f}{\partial y}(1, 0) \left( \frac{-4}{\sqrt{17}} \right) \),

\[
\frac{\partial f}{\partial x}(1, 0) = 2x \bigg|_{(1,0)} = 2
\]

and

\[
\frac{\partial f}{\partial y}(1, 0) = (-6y + 6) \bigg|_{(1,0)} = 6.
\]

So slope \( f_u(1, 0) = \frac{2}{\sqrt{17}} - \frac{6.4}{\sqrt{17}} = -\frac{22}{\sqrt{17}} \).

**The Gradient Vector**

Another way to picture the directional derivative in the direction \( u \)

\[
f_u(a, b) = f_x(a, b)u_1 + f_y(a, b)u_2
\]

is to think of the gradient, or slope, as a vector itself.

\[
\text{grad } f(a, b) = \nabla f(a, b) = f_x(a, b)i + f_y(a, b)j.
\]

For example if \( f(x, y) = x^2 - 3(y - 1)^2 + 3 \)

\[
\nabla f = 2x \mathbf{i} - 6(y - 1) \mathbf{j}
\]

or

\[
\nabla f(1, 0) = 2 \mathbf{i} + 6 \mathbf{j}.
\]

Then the directional derivative is just the dot product of \( \nabla f \) with \( u \) since

\[
\nabla f \cdot u = (f_x \mathbf{i} + f_y \mathbf{j}) \cdot (u_1 \mathbf{i} + u_2 \mathbf{j}) = f_x u_1 + f_y u_2
\]

\[
\Rightarrow f_u(a, b) = \nabla f(a, b) \cdot u
\]

and this is a much easier formula to use.
Example. If \( g(x, y) = e^{x^2} \cos y \) find the directional derivative of \( g(x, y) \) at \((1, \pi)\) in the direction \((-3i + 4j)\).

First we need to find the unit vector in the given direction. Now \( \| -3i + 4j \| = \sqrt{9 + 16} = 5 \). So \( u = -\frac{3}{5} i + \frac{4}{5} j \).

Next find the gradient vector at \((1, \pi)\):

\[
\nabla g = \frac{\partial g}{\partial x} i + \frac{\partial g}{\partial y} j = 2xe^{x^2} \cos y i - e^{x^2} \sin y j
\]

\[
\nabla g(1, \pi) = 2e^1(-1)i - 0j = -2e^1
\]

\[
\nabla g(1, \pi) \cdot (\frac{-3}{5} i + \frac{4}{5} j) = \frac{6e}{5}.
\]

Example. Find the slope of \( f(x, y) = 1 - x^2 - y^2 \) at \((0, 1)\) in the direction \((i - j)\). First \( u = \frac{1}{\sqrt{2}} (i - j) \) is the unit vector in the given direction.

\[
\nabla f = -2xi - 2yj \Rightarrow \nabla f(1, 0) = -2i.
\]

So slope is \( f_u = \nabla f \cdot u = + \frac{2}{\sqrt{2}} = \sqrt{2} \). The gradient vector provides an easy way to calculate the slope in any direction but can we give it a geometrical interpretation?

Consider the contour diagram of a plane \( z = f(x, y) = mx + ny + c \).

The contours \( y = \frac{-m}{n} x + \frac{z_0 - c}{n} \) have slope \( \frac{-m}{n} \) in the \((x, y)\) plane.

The gradient vector, \( mi + nj \), is perpendicular to the contours. Also it points in the direction of increasing \( f \). In fact the direction in which it points is the direction of greatest slope.

But what if \( f(x, y) \) is not a plane.

Consider \( f(x, y) = x^2 + y^2 \). The contours are circles. \( \nabla f = 2xi + 2yj \) which points radially out. Once again \( \nabla f \) points in the direction of greatest slope, perpendicular to the contour lines.

Properties of the Gradient Vector \( \nabla f \)
The direction of $\nabla f(a, b)$ is perpendicular to the contour line through $(a, b)$ and in the direction of increasing $f$. In fact the direction and magnitude of steepest slope at $(a, b)$ is given by $\nabla f(a, b)$.

**Example.** $T(x, y) = 20 - 4x^2 - y^2$ describes the temperature on the surface of a metal plate. $x$ and $y$ are in cm and $T$ is in °C.

In what direction from $(2, -3)$ does the temperature increase most rapidly?

The direction is simply $\nabla T = \frac{\partial T}{\partial x} \hat{i} + \frac{\partial T}{\partial y} \hat{j} = -8x\hat{i} - 2y\hat{j}$. So

$$\nabla T(2, -3) = -16\hat{i} + 6\hat{j}$$

The direction in terms of angles is $\pi - \arctan \left( \frac{6}{16} \right)$.

**Example.** A team of oceanographers are mapping the ocean floor to assist in the recovery of a sunken ship. Using sonar they develop the model

$$D = 250 - 30x^2 - 50\sin \left( \frac{\pi y}{2} \right),$$

where $x$ and $y$ are distance in km, $D$ is depth in meters, and $-2 \leq x \leq 2$ and $-2 \leq y \leq 2$.

(a) Change the model to obtain a graph of the ocean floor.

Let $h(x, y)$ be the height above -250 below sea level. Then $D + h = 250$ and

$$h(x, y) = 30x^2 + 50\sin \left( \frac{\pi y}{2} \right).$$

(b) The ship is located at $(1, 0.5)$. What is its depth?

$$D(1, 0.5) = 250 - 30 - 50\sin \frac{\pi}{4} \approx 184.6 \text{m}.$$

(c) Determine the steepness of the ocean floor in the positive $x$ direction and in the positive $y$ direction. Finally, determine the magnitude and direction of greatest rate of change of depth from the position of the ship.

Slope in the $x$ direction is

$$\frac{\partial h}{\partial x}(1, 0.5) = 60x \bigg|_{(1, 0.5)} = 60.$$
But we must be careful here because \( h \) is in meters while \( x \) and \( y \) are in kilometers. So in fact the slope is \( \frac{60}{1000} = 0.06 \).

Slope in the \( y \) direction is

\[
\frac{\partial h}{\partial y}(1, 0.5) = \frac{50\pi}{2} \cos \frac{\pi y}{2} \bigg|_{(1,0.5)} = 25\pi \cos \frac{\pi}{4} = \frac{25}{\sqrt{2}} \pi.
\]

So slope is \( \frac{25}{\sqrt{2.1000}} \pi = \frac{\pi}{40\sqrt{2}} \).

The direction of greatest rate of change is given by

\[
\nabla h = \frac{60i + \frac{25\pi}{\sqrt{2}}j}{1000}
\]

or \( \arctan \left( \frac{\frac{25\pi}{60\sqrt{2}}}{\frac{5\pi}{12\sqrt{2}}} \right) \) and the magnitude is \( \sqrt{3600 + \frac{625\pi^2}{2}} \).