4. Differential Equations

4.3 Euler’s Method for Solving DE’s Numerically

Euler’s method uses tangent lines as approximations to solution curves. The tangent line to a solution curve of \( \frac{dy}{dt} = f(y, t) \) at \((y_0, t_0)\) can be constructed exactly, without solving for \(y(t)\) because the differential equation itself tells you the slope of the solution curve at \((y_0, t_0)\). The slope is \(f(y_0, t_0)\).

So the tangent line at \((y_0, t_0)\) is

\[
y = y_0 + f(y_0, t_0)(t - t_0)
\]

which approximates the curve for

\[
t \simeq t_0 \text{ or } \Delta t = t - t_0 \text{ small.}
\]

So \(y(t_0 + \Delta t) \simeq y_0 + f(y_0, t_0)\Delta t\). We will call it \(y_1 = y_0 + f(y_0, t_0)\Delta t\), the approximate value of \(y\) after one small step in time.

Now imagine a family of solution curves to the differential equation. We can calculate an approximate value for \(y\) at some later time by taking lots of small steps. At each step we will use the tangent line to the solution pathing through that point.

Let

\[
t_1 = t_0 + \Delta t, \quad t_2 = t_1 + \Delta t, \quad t_3 = t_2 + \Delta t, \quad \ldots, \quad t_n = t_{n-1} + \Delta t.
\]

Take the first step to \((y_1, t_1)\)

\[
y_1 = y_0 + f(y_0, t_0)\Delta t
\]

Now use your approximate value of \(y(t_1) \simeq y_1\), to take the next step.

\[
y_2 = y_1 + f(y_1, t_1)\Delta t \quad \text{then } y_2 \simeq y(t_2).
\]

In general \(y_{n+1} = y_n + f(y_n, t_n)\Delta t\).

**Example.** Use Euler’s Method to find an approximate solution to the initial value problem
\[
\frac{dy}{dt} = 2t, \quad y(0) = 0
\]

for \( y(0.6) \) with \( \Delta t = 0.2 \).

We note that \( f(y_n, t_n) = 2t_n \). Now make the following table:

<table>
<thead>
<tr>
<th>( t_n )</th>
<th>( f(y_n, t_n) )</th>
<th>( y_{n+1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2 \times 0 = 0</td>
<td>0 + 0 \times 0.2 = 0</td>
</tr>
<tr>
<td>0.2</td>
<td>2 \times 0.2 = 0.4</td>
<td>0 + 0.4 \times 0.2 = 0.08</td>
</tr>
<tr>
<td>0.4</td>
<td>2 \times 0.4 = 0.8</td>
<td>0.08 + 0.8 \times 0.2 = 0.24</td>
</tr>
<tr>
<td>0.6</td>
<td>2 \times 0.6 = 1.2</td>
<td>0.24 + 1.2 \times 0.2 = 0.48</td>
</tr>
</tbody>
</table>

So \( y_3 = 0.48 \approx y(0.6) \). The actual value is 0.36!

Of course the method is only accurate if you make \( \Delta t \) small and then the number of steps you have to take is large. So use matlab. Define a \( t \) vector and an initial \( y \) value, then use the for command to iterate. In the example \( \Delta t = 0.05 \) and 12 steps have been taken to take \( t \) to 0.6.

\[
t = (0:0.05:0.6);
\]

\[
y(1) = 0;
\]

for \( i = 1:12 \)

\[
y(i + 1) = y(i) + 2 \times t(i) \times 0.05;
\]

end

\[
y(13)
\]

plot \( (t, y,'-') \)

Using Matlab we find \( y(0.6) = 0.33 \) which is much much better than the previous estimate value for \( y(0.6) \).
**Example.** Consider the differential equation

\[ \frac{dy}{dx} = \sin(xy), \quad \text{with initial condition} \quad y(0) = 1. \]

Estimate \( y(2) \), using Euler’s method with step size \( \Delta x = 0.01 \).

\[
x = (0 : 0.01 : 2);
\]

\[
y(1) = 1;
\]

for \( i = 1 : 200 \)

\[
y(i + 1) = y(i) + \sin(x(i) \cdot y(i)) \cdot 0.01;
\]

end

\[
y(201)
\]

plot \((x, y', -')\)

Using Matlab we find \( y(2) = 1.8243 \).