4. Differential Equations

4.4 Solving Separable Equations

To solve \( \frac{dy}{dt} = t \) we write the equation in differential form

\[
\frac{dy}{dt} = t dt
\]

and integrate

\[
\int dy = \int t dt
\]

\[
\implies y = \frac{t^2}{2} + c.
\]

We can do essentially the same thing with a whole class of more general equations. These are called *separable equations*. For example if \( \frac{dy}{dt} = 2y \) then \( \frac{dy}{y} = 2 dt \). Now integrate

\[
\int \frac{dy}{y} = \int 2 dt
\]

\[
\ln |y| = 2t + c
\]

\[
|y| = e^{2t+c} = e^c e^{2t}
\]

\[
y = Ae^{2t}, \quad \text{where } A = \pm e^c.
\]

In fact any equation that can be put in the form \( \frac{dy}{dt} = \frac{f(t)}{g(y)} \) can be separated, i.e.

\[
g(y)dy = f(t)dt.
\]

Then we integrate.

**Example.** Solve the initial value problem \( \frac{dy}{dt} = te^y \) with \( y(0) = 0 \)

\[
\frac{dy}{e^y} = t dt
\]

\[
\implies \int e^{-y} dy = \int t dt
\]

\[
\implies -e^{-y} = \frac{t^2}{2} + c
\]

\[
\implies y = -\ln \left| -c - \frac{t^2}{2} \right|
\]

\[
y(0) = 0 \implies c = -1. \quad \text{So } y = \ln \left| 1 - \frac{t^2}{2} \right|.
\]

**Quiz:** Which of the following equations are separable?
(a) $\frac{dy}{dt} = y(1 - y)$

(b) $\frac{dy}{dt} = e^{t+y}$

(c) $\frac{dy}{dt} = e^{(t+y)^2}$

(d) $\frac{dy}{dt} = \frac{ty + y}{t^2}$

(e) $\frac{dy}{dt} = ty + y^2$.

**Example.** Solve $\frac{dy}{dt} = \frac{-t}{y}$ and sketch the solution curves in $(t, y)$ space

Separate: $ydy = -tdt$

Integrate: $\frac{y^2}{2} = -\frac{t^2}{2} + c$

$\Rightarrow y^2 + t^2 = 2c$ circles with radius $\sqrt{2c}$. 

![Solution curves for the DE dy/dt = -t/y](image-url)
**Example.** Solve the following initial value problem

\[
\frac{dy}{dx} = \frac{\sin x}{y}, \quad y(0) = 1
\]

\[
\Rightarrow \int y \, dy = \int \sin x \, dx \Rightarrow \frac{y^2}{2} = -\cos x + c
\]

\[
\Rightarrow y = \pm \sqrt{2c - 2\cos x}
\]

\[
y(0) = 1 \Rightarrow 1 = \pm \sqrt{2c - 2}.
\]

This means we must take the + square root and \( c = \frac{3}{2} \).

**Example.** Solve the logistic equation

\[
\frac{dy}{dt} = y(1 - y) \quad \text{with} \quad y(0) = y_0.
\]

Separate

\[
\int \frac{dy}{y(1 - y)} = \int dt.
\]

Now use partial fractions

\[
\frac{1}{y(1 - y)} = \frac{A}{y} + \frac{B}{1 - y} \quad \Rightarrow \quad 1 = A(1 - y) + By \quad \text{for all} \quad y
\]

\[
\Rightarrow 1 = A + (B - A)y.
\]

So \( A = 1 \) and \( B - A = 0 \Rightarrow B = 1 \).

So \( \frac{1}{y(1 - y)} = \frac{1}{y} + \frac{1}{1 - y} \) and the equation becomes

\[
\int \frac{dy}{y} + \int \frac{dy}{1 - y} = \int dt
\]

\[
\Rightarrow \ln|y| - \ln|1 - y| = t + c
\]

\[
\Rightarrow \ln \left| \frac{y}{1 - y} \right| = t + c
\]

\[
\Rightarrow \frac{y}{1 - y} = e^{t + c}
\]

\[
\Rightarrow \frac{y}{1 - y} = Ae^t, \quad \text{where} \quad A = \pm e^c
\]

\[
\Rightarrow y = Ae^t(1 - y)
\]

\[
y(1 + Ae^t) = Ae^t
\]

\[
\Rightarrow y = \frac{Ae^t}{1 + Ae^t}.
\]
Now we use the initial condition to find $A$.

Since
\[ \frac{y}{1 - y} = Ae^t \quad \Rightarrow \quad \frac{y_0}{1 - y_0} = A. \]

So
\[ y = \frac{y_0 e^t}{(1 - y_0) + y_0 e^t} \]

after multiplying top and bottom by $(1 - y_0)$.

Note as $t \to +\infty$ we obtain $y \to 1$. 