5. Parametrisation of Curves and Line Integrals

5.1 Parametrisation of Curves

Circle

Imagine you are in a racing car on a circular track, radius 1 km. Someone in a plane will
see you trace out the circle $x^2 + y^2 = 1$. But to describe your actual motion we need $x(t)$
and $y(t)$.

One parametrisation of the circle would be $x(t) = \cos \omega t$ and $y = \sin \omega t$, which starts at
$(1, 0)$.

Check $(x(t), y(t))$ lies on the circle $x^2 + y^2 = 1$:

$$x^2 + y^2 = \cos^2 \omega t + \sin^2 \omega t = 1.$$  

Then $\omega$ controls the speed you are travelling at.

The circumference of the circle is $2\pi$ and the time it takes to go all the way round is
t = $\frac{2\pi}{\omega}$. So the speed is $\omega$.

But this is not the only parametrisation of the circle. Say you started at $(0, 1)$ and
traversed the circle clockwise. Then

$$\begin{align*}
x(t) &= \cos \omega \left( \frac{\pi}{2\omega} - t \right) \Rightarrow x(t) = \cos \left( \frac{\pi}{2} - \omega t \right) \\
y(t) &= \sin \omega \left( \frac{\pi}{2\omega} - t \right) \Rightarrow y(t) = \sin \left( \frac{\pi}{2} - \omega t \right).
\end{align*}$$

You can use addition formulae or graphs to simplify $x(t)$ and $y(t)$.

Recall

$$\begin{align*}
\sin(a + b) &= \sin a \cos b + \cos a \sin b \\
\cos(a + b) &= \cos a \cos b - \sin a \sin b
\end{align*}$$

So

$$\begin{align*}
x(t) &= \cos \frac{\pi}{2} \cos \omega t + \sin \frac{\pi}{2} \sin \omega t = \sin \omega t \\
y(t) &= \sin \frac{\pi}{2} \cos \omega t - \cos \frac{\pi}{2} \sin \omega t = \cos \omega t.
\end{align*}$$

In general if you start at $(x_0, y_0)$ on a circle with radius $a$ then

$$x(t) = a \cos(\theta \pm \omega t) \quad \text{and} \quad y(t) = a \sin(\theta \pm \omega t),$$
where $\theta = \text{the polar angle} = \arctan\left(\frac{y_0}{x_0}\right)$, and $+$ is for counter clockwise rotation and $-$ is for clockwise rotation.

**Example.** Parametrise a circle centred $(2, 3)$ radius 2, starting at $(0, 3)$ and traversed in a counter clockwise direction.

![Circle Diagram](attachment:circle.png)

The equation of the circle is $(x - 2)^2 + (y - 3)^2 = 4$. So

\[
\begin{align*}
    x - 2 &= 2 \cos(\pi + t) &\Rightarrow& & x &= 2 + 2(\cos \pi \cos t - \sin \pi \sin t) = 2 - 2 \cos t \\
    y - 3 &= 2 \sin(\pi + t) &\Rightarrow& & y &= 3 + 2(\sin \pi \cos t + \cos \pi \sin t) = 3 - 2 \sin t
\end{align*}
\]

**Example.** Parametrise an ellipse centred at $(0, 0)$, crossing the $x$ axis at $\pm 2$ and $y$ axis at $\pm \frac{1}{3}$.

![Ellipse Diagram](attachment:ellipse.png)

The equation of ellipse is $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{\frac{1}{3}}\right)^2 = 1$. 

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Now let $\frac{x}{2} = \cos t$ and $3y = \sin t$. Then

$$x = 2\cos t \quad \text{and} \quad y = \frac{1}{3}\sin t.$$ 

Often curves that cannot be represented as $y = f(x)$ are represented parametrically. (Even the circle $y = \pm\sqrt{1-x^2}$ is strictly speaking not a function as $y$ is not unique.) The classic example is the spiral, which can be thought of as a circle with a variable radius; eg $x(t) = r(t)\cos t$ and $y(t) = r(t)\sin t$.

If $r(t) = e^{-\frac{t}{100}}$ the radius tends to 0 as $t \to +\infty$.

A helix is a circle in $(x, y)$ space, which moves up (and down) in $z$. Say $x(t) = \cos t$, $y(t) = \sin t$, $z(t) = t$. Note none of these parametrisations are unique, $x = \cos \omega t$, $y = \sin \omega t$, $z = \omega t$ parametrises the same helix.

**Parametrisations of straight line paths**
Suppose \( x(t) = \begin{cases} 1 & 0 \leq t \leq 1 \\ 2 - t & 1 \leq t \leq 3 \end{cases} \)

and \( y(t) = \begin{cases} t & 0 \leq t < 2 \\ 2 & 2 \leq t \leq 3 \end{cases} \)

Then

for \( 0 \leq t < 1 \quad x(t) = 1 \) and \( y(t) = t \Rightarrow x = 1 \)

for \( 1 \leq t \leq 2 \quad x(t) = 2 - t \) and \( y(t) = t \Rightarrow y = 2 - x \)

for \( 2 \leq t \leq 3 \quad x(t) = 2 - t \) and \( y(t) = 2 \Rightarrow y = 2 \)

If \( x(t) \) and \( y(t) \) are composed of straight line segments then you can find a parametrisation so that \( x(t) \) and \( y(t) \) are too.

Parametrise the following curve:
For $t = 0$ we have $x(0) = 0$ and $y(0) = -2$.

For $0 \leq t < 1$ we have $y = -x - 2$. So let $x = -t$ then $y = t - 2$.

For $1 \leq t < 2$ we have $x = -1$ and $y = -1 + (t - 1) = -2 + t$.

For $2 \leq t \leq 3$ we have $y = x + 1$. So let $x = -1 + (t - 2) = t - 3$ then $y = t - 2$.

Therefore,

$$x = \begin{cases} 
-t & \text{for } 0 \leq t < 1 \\
-1 & \text{for } 1 \leq t < 2 \\
t - 3 & \text{for } 2 \leq t \leq 3 
\end{cases} \quad \text{and} \quad y = t - 2 \quad \text{for } 0 \leq t \leq 3.$$

Lines can be parametrised systematically using vectors. For example $y = 1 - x$ passes through $(0, 1)$ and is parallel to the vector $\hat{i} - \hat{j}$.

To parametrise the line let $\hat{r}_0 = 0\hat{i} + \hat{j}$ be the position vector of the point $(0, 1)$. Then any point on the line $y = 1 - x$ is given by a vector

$$\hat{r} = \hat{r}_0 + t(\hat{i} - \hat{j}) \quad \text{for some } t \quad \Rightarrow \quad \hat{r} = \hat{j} + t(\hat{i} - \hat{j}) = t\hat{i} + (1 - t)\hat{j}.$$

In general if $\hat{r}_0$ is the position vector of a point on the line and $\hat{v}$ is a vector parallel to the line any point on the line is given by

$$\hat{r} = \hat{r}_0 + t\hat{v} \quad \text{for some } t.$$

**Example.** Find the intersection point of the line, passing through $(0, 1, 0)$ and parallel to the vector $\hat{v} = 2\hat{i} - \hat{j} + \hat{k}$, with the hyperboloid

$$x^2 + \left(\frac{y}{2}\right)^2 - x^2 = 1.$$

First parametrise the line using vectors $r_0 = \hat{j}$ and $\hat{v}$. Any point on the line is given by

$$\hat{r} = \hat{r}_0 + t\hat{v} \quad \Rightarrow \quad \hat{r} = \hat{j} + t(2\hat{i} - \hat{j} + \hat{k}) = 2t\hat{i} + (1 - t)\hat{j} + t\hat{k}.$$

So $x = 2t$, $y = 1 - t$ and $z = t$. Now the intersection points with the hyperboloid can be found as follows:

$$(2t)^2 + \left(\frac{1 - t}{2}\right)^2 - t^2 = 1 \Rightarrow 13t^2 - 2t - 3 = 0 \Rightarrow t = \frac{1 \pm 2\sqrt{10}}{13} \quad \text{two intersection points.}$$