MATH 2644, Review for Hour Exam 4

- The exam will be on the Sections 11.1–11.9.

- If you are late for the exam, I interpret that you do so at your own risk. There won't be any way to make up your lost time.

- Remember the policy in case you fall sick: You must write me an email BEFORE the exam — no excuse! If you fail to do so, there will be a penalty.

1. §11.1. Sequences \( \{a_n\} \), \( \lim_{n \to \infty} a_n \)
   - Limit Laws
   - L'Hospital's rule
   - \( r \)-test
   - Squeeze theorem; general version
   - Squeeze theorem; weak version
   - Continuous-(tres)passing theorem
   - Monotone bounded sequence theorem

2. §11.2–11.7 Series \( \sum_{n=k}^{\infty} a_n \) (Sections 11.3–11.6)
   - Geometric series
   - Telescoping series
   - Harmonic series
   - Test for Divergence (via \( \lim a_n \))
   - Limit Laws for series
   - Integral test
   - \( p \)-test
   - Comparison test
   - Limit Comparison test
   - Alternating Series test
   - Absolute convergence vs. Conditional convergence
   - Absolute convergence theorem
   - Ratio test (for Absolute convergence)
   - Root test (for Absolute convergence)

3. §11.8–11.9 Power Series
   - Interval of Convergence, Radius of Convergence
   - Power series expansion of functions: As written on the Class Log on our course webpage, you will be asked simpler ones such as, among the examples done in class, a power series expansion of \( 1/x \) at \( x = 1 \) or Examples 1,2 in §11.9. (Among the examples done in class, more difficult ones like Examples 3,6 in §11.9 will not be on the exam.)
Do not expect that the problems of these review problems will be almost identical to the problems of the actual exam.

About 70% of the problems will have tests/theorems given to use; 20% will not because they will use very basic tests that were not boxed in the list (such as r-test, p-test, or geometric test); One problem (10%) will need a “sophisticated method” yet the name of the methods will be not be given in order to appreciate true A.

Specify the names of the theorems/tests that you use.

You must check and state ALL the conditions before you apply tests. This is particularly important because the best methods will be given to use rather than you come up with the methods.

Most of the following problems were either on a quiz or presented in class.

Warm-up: Before start the real problems, let’s check one common mistake that you seemed to remember well at one point but appear to have forgotten:

(i) State the Test for Divergence for series.

(ii) Is the following statement true?

"If the sequence $\lim_{n \to \infty} a_n = 0$, then the series $\sum_{n=1}^{\infty} a_n$ converges."

Yes, it is false. Give a counter-example of the statement.
Specify the names of the theorems/tests that you use. You must check and state ALL the conditions before you apply theorems/tests.

1. For the given sequence \( \left\{ \frac{3^n}{5^n} \right\} \), \( n = 0, 1, 2, \ldots \)

(a) Expand the sequence by writing at least the first three terms and the \( n \)th term.

\[
1, \quad \frac{2}{5}, \quad \left(\frac{2}{5}\right)^2, \quad \ldots, \quad \left(\frac{3}{5}\right)^{n-1}, \quad \left(\frac{3}{5}\right)^n,
\]

(b) Determine whether the sequence converges or diverges. If converges, find the limit value. \( \text{Quiz 2} \)

2. For the given sequence \( \left\{ \frac{(-1)^n}{n} \right\} \), \( n = 1, 2, 3, \ldots \)

(a) Expand the sequence by writing at least the first three terms and the \( n \)th term.

\[
-\frac{1}{1}, \quad \frac{1}{2}, \quad -\frac{1}{3}, \quad \ldots, \quad (-1)^n \cdot \frac{1}{n}, \ldots
\]

(b) Determine whether the sequence converges or diverges. If converges, find the limit value. (Hint: Both the general squeeze theorem and the weak squeeze theorem work though I think the weak version would be simpler.) \( \text{Quiz 2} \)

**Method 1 (Weak Sq. Thm.)**

\[
a_n = \frac{(-1)^n}{n}, \quad |a_n| = \frac{1}{n}, \quad \lim_{n \to \infty} |a_n| = \frac{1}{n} = 0.
\]

By weak Sq. Thm., \( \lim_{n \to \infty} \frac{(-1)^n}{n} = 0. \)

**Method 2 (General Sq. Thm.)**

\[
-\frac{1}{n} \leq \frac{(-1)^n}{n} \leq \frac{1}{n} \quad \text{for all} \ n = 1, 2, \ldots
\]

Since \( \lim_{n \to \infty} \left(\frac{-1}{n}\right) = \lim_{n \to \infty} \left(\frac{1}{n}\right) = 0 \), by Sq. Thm.,

\[
\lim_{n \to \infty} \frac{(-1)^n}{n} = 0.
\]
Specify the names of the theorems/tests that you use. You must check and state ALL the conditions before you apply theorems/tests.

3. A sequence \{a_n\} is defined in a recurrence relation by

\[ a_1 = 1, \quad a_{n+1} = \frac{2}{3} a_n \quad n = 1, 2, 3, \ldots \]

(a) Expand the sequence by writing at least the first three terms and the \(n\)th term.

\[-1, \frac{2}{3}, \left(\frac{2}{3}\right), \ldots, \left(\frac{2}{3}\right)^{n-1}, \left(\frac{2}{3}\right)^n, \ldots \]

(b) Determine whether the sequence \{a_n\} is monotone increasing or decreasing or neither.

(c) Show that the sequence \{a_n\} is bounded.

(d) Determine whether the sequence converges or not. If converges, give its limit value.

By Monotone-bounded Seq. Thm, \(\sum a_n\) is convergent.

Setting \(s = \lim a_n\) and applying limit laws,

\[ s = \frac{2}{3} s \Rightarrow \frac{1}{3} s = 0 \Rightarrow s = 0. \]
Specify the names of the theorems/tests that you use. You must check and state ALL the conditions before you apply theorems/tests.

4. Determine whether the following sequences are monotone increasing or decreasing or neither. (You will need their monotone properties later on.)

Hint: Show one of the followings, whichever you prefer:

- \( a_n \geq a_{n+1} \) if decreasing, or \( a_n \leq a_{n+1} \) if increasing, for \( n \geq 1 \) or 2, whichever appropriate.
- For \( f(x) \) that is the “function version” of \( a_n \), show that the first derivative \( f'(x) < 0 \) if decreasing, or \( f'(x) > 0 \) if increasing for \( x \geq 1 \) or 2, whichever appropriate.

(I would go for the first option for (a)(b) and the second option for (c)(d)(e). However, you yourself should try both options to see which one is better for what types in order to make it yours.)

(a) \[ \left\{ \frac{1}{n \ln n} \right\} \text{ for } n \geq 2. \]

\[
\frac{1}{n \ln n} \geq \frac{1}{(n+1) \ln(n+1)} \quad \text{for } n \geq 2
\]

\[ \Rightarrow a_{n-1} > a_n \text{ for } n \geq 2 \]

\[ \Rightarrow \{a_n\} \text{ monotone decreasing} \]

(b) \[ \left\{ \frac{n+1}{n^2} \right\} \text{ for } n \geq 1. \]

\[
a_n = \frac{n+1}{n^2} = \frac{1}{n} + \frac{1}{n^2} \geq \frac{1}{n+1} + \frac{1}{(n+1)^2} = a_{n+1}
\]

\[ \frac{1}{n} \geq \frac{1}{n+1} \text{ and } \frac{1}{n^2} \geq \frac{1}{(n+1)^2} \quad \text{for } n \geq 1.
\]

\[ \Rightarrow \{a_n\} \text{ monotone decreasing} \]
(c) \[ \left\{ \frac{n}{\sqrt[n]{n^5 + 1}} \right\} \text{ for } n \geq 1 \]

\[ f(x) = \frac{x}{\sqrt[n]{x^5 + 1}}, \quad x > 1. \]
\[ f'(x) = \frac{\frac{5}{x^5 + 1} - x - \frac{1}{2}(x^5 + 1)^{-\frac{1}{2}} \cdot 5x^4}{x^5 + 1} = \frac{(x^5 + 1) - \frac{5}{2}x^5}{(x^5 + 1)\sqrt{x^5 + 1}} = \frac{2x^5 + 2 - 5x^5}{2(x^5 + 1)\sqrt{x^5 + 1}} = \frac{2 - 3x^5}{2(x^5 + 1)\sqrt{x^5 + 1}}. \]

(d) \[ \left\{ \frac{\ln n}{n} \right\} \text{ for } n \geq 2. \]

\[ f(x) = \frac{\ln x}{x}, \quad x > 2. \]
\[ f'(x) = \frac{\frac{1}{x} \cdot x - \ln x}{x^2} = \frac{1 - \ln x}{x^2} \leq 0 \text{ for } x > 3. \]

\[ \Rightarrow f(x) \text{ is } \overline{\text{decrease}} \text{ for } x > 3 \text{ and so is } \left\{ \frac{\ln n}{n} \right\} \text{ for } n > 3. \]

(e) \[ \left\{ \frac{n}{\ln n} \right\} \text{ for } n \geq 1. \]

\[ f(x) = \frac{x}{e^x} = xe^{-x}, \quad x > 1 \]
\[ f'(x) = e^{-x} + xe^{-x}(1) = e^{-x} - xe^{-x} = e^{-x}(1-x) \leq 0 \text{ for } x > 1. \]

\[ \Rightarrow e^x > 0 \quad \text{for } x > 1 \]
\[ \Rightarrow f(x) \text{ is } \overline{\text{decrease}} \text{ for } x > 1 \text{ and so is } \left\{ \frac{n}{\ln n} \right\}. \]
Specify the names of the theorems/tests that you use. You must check and state ALL the conditions before you apply theorems/tests.

5. Determine whether the following series $\sum_{n=1}^{\infty} a_n$ converge or diverge. If converge, find the limit values.

(a) $\sum_{n=1}^{\infty} \frac{n^2}{100n^2 + 1}$

(b) $\sum_{n=0}^{\infty} 2^n 3^{-n}$
Specify the names of the theorems/tests that you use. You must check and state ALL the conditions before you apply theorems/tests.

6. For the given series \( \sum_{n=2}^{\infty} \frac{1}{n \ln n} \),

(a) Expand the series by writing at least the first three terms and the \( n \)th term.

(b) Use (either Test for Divergence or) Integral test to determine whether the given series converges or diverges.
Specify the names of the theorems/tests that you use. You must check and state ALL the conditions before you apply theorems/tests.

7. For the given series \( \sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n} \),

(a) Expand the series by writing at least the first three terms and the \( n \)th term.

\[
\{ \frac{1}{2 \ln 2} - \frac{1}{3 \ln 3} + \frac{1}{4 \ln 4} \ldots + (-1)^n \frac{1}{n \ln n} + (-1)^{n+1} \frac{1}{(n+1) \ln (n+1)} \uparrow \}
\]

(b) Use (either Test for Divergence or) Alternating series test to determine whether the given series converges or diverges.

1. alternating series

\[\frac{1}{n \ln n} \geq \frac{1}{(n+1) \ln (n+1)}\]

\(\{a_n = 1 \quad a_{n+1} = \frac{1}{n \ln n} \}

2. \[\lim_{n \to \infty} \frac{a_n}{a_{n+1}} = \lim_{n \to \infty} \frac{\ln n}{\ln (n+1)} = 0\]

By Alternating series test, \( \sum_{n=2}^{\infty} (-1)^n \frac{1}{n \ln n} \) converges.

(c) The given series

i. converges absolutely

\[\square\]

ii. converges conditionally

\[\square\]

iii. diverges
Specify the names of the theorems/tests that you use. You must check and state ALL the conditions before you apply theorems/tests.

8. For the given series \( \sum_{n=1}^{\infty} \frac{\ln n}{n} \),

(a) Expand the series by writing at least the first three terms and the \( n \)th term.

\[
\frac{\ln 1}{1} + \frac{\ln 2}{2} + \frac{\ln 3}{3} + \ldots + \frac{\ln n}{n} + \ldots
\]

(b) Use (either Test for Divergence or) Integral test to determine whether the given series converges or diverges.

\[ f(x) = \frac{\ln x}{x}, \quad x \geq 1 \]

1. \( f(x) \) nonnegative for \( x \geq 1 \)
2. \( f(x) \) decreasing. \( \frac{d}{dx} (\ln x / x) = \frac{1}{x^2} \left( \ln x - 1 \right) \) by #4(d)
3. \( f(x) \) continuous for \( x \geq 1 \)
4. \( \int_{1}^{\infty} \frac{\ln x}{x} \, dx = \lim_{b \to \infty} \int_{1}^{b} \frac{\ln x}{x} \, dx = \lim_{b \to \infty} \left[ u \ln u - \frac{u}{2} \right]_{1}^{b} = \lim_{b \to \infty} \left( \ln b - \frac{b}{2} \right) = \infty \) diverges

(c) Use (either Test for Divergence or) Comparison test to determine whether the given series converges or diverges.

\[ \sum_{n=1}^{\infty} \frac{\ln n}{n} \text{ diverges} \]