Specify the names of the theorems/tests that you use. You must check and state ALL the conditions before you apply theorems/tests.

9. For the given series \( \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\ln n}{n} \), \( \text{Similarly to \#7.} \)

(a) Expand the series by writing at least the first three terms and the \( n \)th term.

(b) Use (either Test for Divergence or) Alternating series test to determine whether the given series converges or diverges.

(c) The given series
   i. converges absolutely
   ii. converges conditionally
   iii. diverges
Specify the names of the theorems/tests that you use. You must check and state ALL the conditions before you apply theorems/tests.

10. For the given series \( \sum_{n=1}^{\infty} \frac{n}{\sqrt{n^5 + 1}} \),

(a) Expand the series by writing at least the first three terms and the \( n \)th term. \textit{On your own}

(b) Use (either Test for Divergence or) Comparison test to determine whether the given series converges or diverges.

\[
\frac{n}{\sqrt{n^5 + 1}} \sim \frac{n}{n^{5/2}} = \frac{1}{n^{3/2}} \text{ for } n > 1
\]

- \( 0 \leq \frac{n}{\sqrt{n^5 + 1}} \leq \frac{n}{\sqrt{n^5}} = \frac{1}{n^{3/2}} \text{ converges by } p \text{-test with } p = \frac{3}{2} \)

By Comp.test, \( \sum \frac{n}{\sqrt{n+1}} \) converges

(c) Use (either Test for Divergence or) Limit Comparison test to determine whether the given series converges or diverges.

\[
\frac{a_n}{a_m} = \frac{\frac{n}{\sqrt{n^5 + 1}}}{\frac{m}{\sqrt{m^5 + 1}}} = \frac{n \cdot \sqrt{m}}{\sqrt{n^5 + 1}} = \frac{\sqrt{n^5}}{1 + \frac{1}{n^5}}
\]

\( n \to \infty \)

\( \frac{1}{1 + \frac{1}{n^5}} \to 1 \)

- \( \frac{n}{\sqrt{n^5 + 1}} \)

- \( c = 1 > 0 \)

\( (\text{ie, nonzero positive constant}) \)

- \( \sum \frac{\frac{n}{\sqrt{n^5 + 1}}}{\frac{1}{\sqrt{n^{5/2}}}} \) converges by \( p \)-test with \( p = \frac{3}{2} \)

By Limit Comp.test, \( \sum a_n \) converges.
Specify the names of the theorems/tests that you use. You must check and state ALL the conditions before you apply theorems/tests.

11. For the given series \( \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{\sqrt{n^5 + 1}} \), \( a_n = (-1)^{n+1} \frac{n}{\sqrt{n^5 + 1}} \)

(a) Expand the series by writing at least the first three terms and the \( n \)th term.  

(b) Use (either Test for Divergence or) the result of #10 and Absolute Convergent Theorem to determine whether the given series converges or diverges.

\[ \sum |a_n| = \sum \frac{n}{\sqrt{n^5 + 1}} \] converges from #10.

By Absolute Conv. Thm, 

\[ \sum a_n \] converges absolutely.

(c) Use (either Test for Divergence or) Alternating series test to determine whether the given series converges or diverges.  

on your own

(d) The series \( \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{\sqrt{n^5 + 1}} \)

i. converges absolutely  
ii. converges conditionally  
iii. diverges

(Think point: Do you think you had to do #11c in order to answer this question?)
Specify the names of the theorems/tests that you use. You must check and state ALL the conditions before you apply theorems/tests.

12. For the given series \( \sum_{n=1}^{\infty} \frac{n}{e^n} \)

(a) Expand the series by writing at least the first three terms and the \( n \)th term.

(b) Use (either Test for Divergence or) Ratio test to determine whether the given series converges or diverges.

(c) Use (either Test for Divergence or) Integral test to determine whether the given series converges or diverges.

\[ f(x) = \frac{x}{e^x}, \quad x \geq 1 \]

1. \( f(x) > 0 \) for \( x \geq 1 \)

2. \( f(x) \) decreases for \( x \geq 1 \) by \#4(e)

3. \( f(x) \) is continuous for \( x \geq 1 \)

\[ \int_{1}^{\infty} \frac{x}{e^x} \, dx = \lim_{t \to \infty} \int_{1}^{t} \frac{x}{e^x} \, dx = \lim_{t \to \infty} \left[ x e^{-x} + \int e^{-x} \, dx \right]_{1}^{t} \]

\[ = \lim_{t \to \infty} \left( [1 e^{-1}] + [e^{-t}] \right) \]

\[ = \lim_{t \to \infty} \left( 1 e^{-1} + e^{-t} \right) \Rightarrow \text{Int. test,} \]

\[ \sum_{n=1}^{\infty} \frac{n}{e^n} \text{ converges.} \]
Specify the names of the theorems/tests that you use. You must check and state ALL the conditions before you apply theorems/tests.

13. For the given series \( \sum_{n=1}^{\infty} (-1)^n \frac{n}{e^n} \),

(a) Expand the series by writing at least the first three terms and the \( n \)th term. On your own.

(b) Use (either Test for Divergence or) any methods/theorem that you like to determine whether the given series converges or diverges.

You could do, again, Alt. series test.
But we've already shown \( \sum |a_n| \) converges.
Hence \( \sum a_n \) converges absolutely.

(c) The series \( \sum_{n=1}^{\infty} (-1)^n \frac{n}{e^n} \)

- converges absolutely
- converges conditionally
- diverges

(Think point: Do you think you had to do #13b in order to answer this question?)
Specify the names of the theorems/tests that you use. You must check and state ALL the conditions before you apply theorems/tests.

14. For the given series \( \sum_{n=1}^{\infty} \frac{n^n}{n!} \), Lecture Notes

(a) Expand the series by writing at least the first three terms and the \( n \)th term.

(b) Use (either Test for Divergence or) Ratio test to determine whether the given series converges or diverges.
Specify the names of the theorems/tests that you use. You must check and state ALL the conditions before you apply theorems/tests.

15. For the given series \( \sum_{n=1}^{\infty} \frac{n + 1}{n^2} \), determine whether it is absolutely convergent, conditionally convergent, or divergent. Refer to the last quiz.
Specify the names of the theorems/tests that you use. You must check and state ALL the conditions before you apply theorems/tests.

16. For a given power series $\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$ (this is Example 5 in Section 11.8)

(a) Expand the power series by writing at least the first three terms and the $n$th term.

$$0 + \frac{(x+2)^2}{3^2} + \frac{2(x+2)^3}{3^3} + \cdots + \frac{(n-1)(x+2)^{n-1}}{3^n} + \frac{n(x+2)^n}{3^{n+1}} + \cdots$$

(b) Use Ratio test to find the interval of convergence $I$ and the radius of convergence $R$.

\[
\left|\frac{a_{n+1}}{a_n}\right| = \left| \frac{n(x+2)^n}{3^{n+1}} \right| \left| \frac{(n-1)(x+2)^{n-1}}{3^n} \right| = \left| \frac{n}{(n-1)} \frac{(x+2)^n}{3^{n+1}} \frac{(x+2)^{n-1}}{3^n} \right| = \frac{1}{1-\frac{1}{3}} \cdot \frac{1}{3} \cdot |x+2| \quad \text{as} \quad h \to \infty
\]

\[
L = \frac{1}{3} |x+2| < 1 \quad \iff \quad |x+2| < 3
\]

The power series converges absolutely.

\[
L = \frac{1}{3} |x+2| > 1 \quad \iff \quad |x+2| > 3
\]

The power series diverges.

\[
L = \frac{1}{3} = 1 \quad \iff \quad x = -5 \text{ or } x = 1
\]

$x = -5 \iff \sum \frac{n(-3)^n}{3^{n+1}} = \sum (-1)^n \frac{n3^n}{3^{n+1}} = \sum \frac{(-1)^n}{3} \text{ diverges}$

$x = 1 \iff \sum \frac{n3^n}{3^{n+1}} = \sum \frac{n}{3} \text{ diverges}$ by Test for Divergence (i.e., $\lim_{n \to \infty} \frac{n}{3} \neq 0$)

$I = (-5, 1), \quad R = 3$
Specify the names of the theorems/tests that you use. You must check and state ALL the conditions before you apply theorems/tests.

17. For a given function $f(x) = \frac{1}{1 + x^2}$, (this is Example 1 in Section 11.9)

(a) Use geometric series to represent the given function as a power series at $x = 0$.

$$f(x) = \frac{1}{1 - (-x^2)} = (-x^2)^0 + (-x^2)^1 + (-x^2)^2 + \cdots$$

$$= \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

(b) What is the range of $x$ where the power series expansion in (a) is valid.

$$\left|-x^2\right| < 1 \iff \left|x^2\right| < 1$$

$$\iff \left|x\right| < 1$$

As usual, redo the examples done in class, and then quizzes. That is the best way to study.