Sample Research Problems:

Note: These may not coincide with actual problems that we will be encounter in summer 2014, but they are certainly representative of the types of problems that we will be working on.

A. Mixed van der Waerden Numbers. The classical van der Waerden number, \( w(k;r) \), is defined as the least positive integer \( n \) such that for every partition of \( \{1,2,...,n\} \) into \( r \) subsets, there is some subset containing a \( k \)-term arithmetic progression (a.p.). Using the language of “colorings,” \( w(k;r) \) is defined as the least positive integer \( n \) such that every \( r \)-coloring of \( \{1,2,...,n\} \) admits a monochromatic \( k \)-term a.p. Slightly generalizing this concept, the mixed van der Waerden number \( w(k_0,k_1,...,k_{r-1};r) \) is defined as the least positive integer \( n \) such that for every \( r \)-coloring \( \chi : 1,2,...,n \rightarrow \{0,1,...,r-1\} \) there is, for some \( i, 0 \leq i \leq r-1 \), a \( k_i \)-term a.p. of color \( i \). Although the classical van der Waerden numbers \( w(k;r) \) have received much attention for almost eighty years (with many of the fundamental questions still unanswered), relatively little attention has been given to the mixed (or “off-diagonal”) van der Waerden numbers when compared to, say, the classical (mixed) graph-theoretical Ramsey numbers \( R(k_1,k_2,...,k_r) \). In recent work by Culver, Landman, and Robertson [CLR], by Khodkar and Landman [KL], and Ahmed [A], several new exact values of the mixed van der Waerden numbers have been found. In particular, for the special case of \( w(k,2,2,...,2;r) \), exact formulas are given in terms of \( k \) and \( r \). There are several intriguing, but probably not extremely difficult, questions remaining. One such question, which seems a natural “next step,” is to find reasonably good bound(s) (upper or lower) on \( w(3,3,2,2,...,2;r) \). A bit of a generalization of this problem would be to get some information on the magnitude of \( w(3,3,...,3,2,2,...,2;r) \); put another way, we want to study the magnitude of \( n(\ell,m) = w(3,3,...,3,2,2,...,2;r) \), where there are \( \ell \) 3’s and \( m \) 2’s.

B. Ramsey Numbers for Quasi-Progressions. Reasonable upper bounds on the van der Waerden numbers \( w(k;r) \) have been notoriously difficult to produce. If we substitute, in the definition of \( w, \) a much “larger” family of sequences than the family, \( AP, \) of a.p.’s, then this will produce smaller numbers for the associated “van der Waerden-type” numbers (often referred to as the associated Ramsey numbers for the family of sequences being considered). One way to generalize the notion of an a.p. is by what Erdős dubbed a “quasi-progression.” The definition is as follows: a \( k \)-term quasi-progression of diameter \( n \), where \( n \geq 0 \) and \( k \geq 2 \), is a sequence of positive integers \( \{x_1,x_2,...,x_k\} \) such that there exists a positive integer \( d \) with the property that \( d \leq x_i - x_{i-1} \leq d + n \) for \( i = 2,3,...,k \). Denote by \( Q_n(k;r) \) the associated Ramsey function for quasi-progressions of diameter \( n \). Landman [L1] provides exact
formulas for $Q_n(n + 1; 2)$ (an easy case) and $Q_n(n + 2; 2)$, and also gives a polynomial upper bound, in terms of $n$ and $k$, for $Q_n(k; 2)$ provided $k \leq 3n/2$. Three wide-open problems are: (i) finding an upper bound for $Q_n(k; 2)$ for (any) values of $k$ that are greater than $3n/2$; (ii) what can be said about the magnitude of the function $Q_n(k; 3)$?; and (iii) finding an exact formula for $Q_n(n + 3; 2)$.

**C. Schur Numbers.** The Schur number $s(r)$, $r \geq 1$, is the least positive integer such that for every $r$-coloring of $\{1, 2, \ldots, s(r)\}$ there is a monochromatic solution to $x + y = z$. Among the values known for the Schur numbers are $s(1) = 2$, $s(2) = 5$, $s(3) = 14$, and $s(4) = 45$. An open conjecture is that $s(5) = 160$. There are also many refinements and offshoots of the Schur function $s(r)$ that can be explored.

**D. Regularity of $(a,b)$-triples.** For positive integers $a \leq b$, an $(a,b)$-triple is a set $\{x, ax + d, bx + 2d\}$ where $x, d \in \mathbb{Z}^+$. Denote by $n = n(a,b;r)$ the least positive integer (if it exists) such that every $r$-coloring of $\{1, 2, \ldots, n\}$ yields a monochromatic $(a,b)$-triple. We say that $(a,b)$ is regular if $n(a,b;r)$ exists (is finite) for all $r \geq 1$. Obviously, the $(1,1)$-triples are just the $3$-term a.p.’s, so by van der Waerden’s theorem, $(1,1)$ is regular. As it turns out, $(1,1)$ is the only regular pair. For $(a,b) \neq (1,1)$, the degree of regularity of $(a,b)$, $\text{dor}(a,b)$, is the greatest integer $r$ such that $n(a,b;r)$ exists. Several interesting questions remain unanswered here. In particular, the only pair $(a,b) \neq (1,1)$ known (thus far) for which $\text{dor}(a,b) > 2$ is $(2,2)$ (Frantzikinakis, Landman, and Robertson [FLR] found that $\text{dor}(2,2) = 3$). Are there other such pairs? Are there infinitely many? Are there any not of the form $(a,2a-2)$?

**E. Progressions with Tails.** Another type of sequence for which the a.p.’s are a special case are the “augmented arithmetic progressions with a tail.” More explicitly, define $g_b(b;r)$ to be the least positive integer $n$ such that every $r$-coloring of $\{1, 2, \ldots, n\}$ admits a monochromatic sequence of the form $\{x, x+d, x+2d, \ldots, x+(k-2)d, x+(k-1)d+b\}$ ($b$ is the tail). Grynkiewicz [G1] has shown that $g_3(b;2) = 2b+10$ for all even $b \geq 10$ (it is a simple exercise to show that $g_3(b;2)$ does not exist when $b$ is odd). Very little is known if $k > 3$ or $r > 2$. Perhaps the first line of questioning that would prove fruitful is the exploration of $g_6(4;2)$. Bialostocki, Leffmann, and Meerdink [BLM] have shown that $g_3(b;8)$ never exists, and that $g_3(b;3) \leq \frac{55}{6}b + 1$ for $b$ a multiple of 6. There is a good chance that the methods employed in [R2] or [G1], which successively improved the bounds for $g_3(b;2)$ given in [BLM], will give rise to an improved upper bound on $g_3(b;3)$.

**F. Super edge-graceful labeling in graphs:** Define an edge labeling as a bijection

$$f : E(G) \rightarrow \{0, \pm 1, \pm 2, \ldots, \pm \frac{q-1}{2}\}$$

for $q$ odd

or

$$f : E(G) \rightarrow \{\pm 1, \pm 2, \ldots, \pm \frac{q}{2}\}$$

for $q$ even.

For every vertex $x \in V(G)$, define the induced vertex labeling of $x$ as $f^*(x) =$
\[ \sum_{xy \in E(G)} f(xy). \]

If

\[ f^* : V(G) \to \{0, \pm 1, \pm 2, \ldots, \pm \frac{p-1}{2} \} \text{ for } p \text{ odd} \]

or

\[ f^* : V(G) \to \{\pm 1, \pm 2, \ldots, \pm p \} \text{ for } p \text{ even}, \]

is a bijection, then the labeling \( f \) is super edge-graceful.

Super edge-graceful labelings were first considered in [MS], and it was shown that super edge-graceful trees are edge-graceful. In particular, if \( G \) is a super-edge graceful graph and

\[
\begin{align*}
    p & | q & \text{ if } q \text{ is odd} \\
    p & | q + 1 & \text{ if } q \text{ is even},
\end{align*}
\]

then \( G \) is edge-graceful.

Some families of graphs have been shown to be super-edge graceful by explicit labelings. For example, it is known that paths of all orders except 2 and 4 and cycles of all orders except 4 and 6 are super edge-graceful [CFKX], as are trees of odd order with three even vertices [LH], complete graphs of all orders except 1, 2 and 4 [KRS], complete bipartite graphs except \( K_{2,2}, K_{2,3}, \) and \( K_{1,n} \) if \( n \) is odd [KNP] [REU 2008], complete tripartite graphs except \( K_{1,1,2} \) [K], and total stars and total cycles [KV] [REU 2009].

We do not know whether many graphs are super edge-graceful. Researchers are interested in knowing whether the following is true:

**Conjecture 2.** [Le] All odd-order trees are super edge-graceful.

\[ G. \textbf{Signed edge domination in graphs:} \]

Let \( G \) be a graph with vertex set \( V(G) \) and edge set \( E(G) \). Two edges \( e_1, e_2 \) of \( G \) are called adjacent if they are distinct and have a common end-vertex. The open neighborhood \( N_G(e) \) of an edge \( e \in E(G) \) is the set of all edges adjacent to \( e \). Its closed neighborhood is \( N_G[e] = N_G(e) \cup \{e\} \). For a function \( f : E(G) \to \{-1, 1\} \) and a subset \( S \) of \( E(G) \), we define \( f(S) = \sum_{e \in S} f(e) \).

For each vertex \( v \in V(G) \), we also define \( f(v) = \sum_{e \in E(v)} f(e) \), where \( E(v) \) is the set of all edges at vertex \( v \). A function \( f : E(G) \to \{-1, 1\} \) is called a signed edge dominating function of \( G \), if \( f(N_G[e]) \geq 1 \) for each edge \( e \in E(G) \). The minimum of the values of \( w(f) = f(E(G)) \), taken over all signed edge dominating functions \( f \) of \( G \), is called the signed edge domination number of \( G \). The signed edge domination number was introduced in [X1] and denoted by \( \gamma'_s(G) \).

**Conjecture 3.** [X2] For all graphs \( G \) of order \( n \), \( \gamma'_s(G) \leq n - 1 \).

In [X2] it was proved that for all graphs \( G \) of order \( n \), \( \gamma'_s(G) \leq \lfloor \frac{5n}{3} n - 1 \rfloor \). An improved upper bound can be found in [KKS]; that is, for any simple graph \( G \) of order \( n \), \( \gamma'_s(G) \leq \lceil \frac{3n}{2} \rceil \).
H. Twofold 2-perfect 8-cycle systems with an extra property: A twofold 8-cycle system is an edge-disjoint decomposition of a twofold complete graph (which has two edges between every pair of vertices) into 8-cycles. The order of the complete graph is also called the order of the 8-cycle system. A twofold 2-perfect 8-cycle system is a twofold 8-cycle system such that the collection of distance 2 edges in each 8-cycle also covers the complete graph, forming a (twofold) 4-cycle system. Existence of 2-perfect 8-cycle systems for all admissible orders was shown in [AB], although $\lambda$-fold existence for $\lambda > 1$ has not been done.

In [BK], Khodkar and Billington imposed an extra condition on the twofold 2-perfect 8-cycle system. They required that the two paths of length two between each pair of vertices, say $x, a_{xy}, y$ and $x, b_{xy}, y$, should be distinct, that is, with $a_{xy} \neq b_{xy}$; thus, they form a 4-cycle $(x, a, y, b)$. The authors completely solved the existence of such twofold 2-perfect 8-cycle systems with this “extra” property. All admissible orders congruent to 0 or 1 modulo 8 can be achieved, apart from order 8. The next open case is twofold 2-perfect 10-cycle systems with an extra property.

I. Super-simple twofold $m$-cycle systems: A $(v, k, \lambda)$-block design is an ordered pair $(V, B)$, where $V$ is a set of size $v$ and $B$ is a collection of $k$-subsets (called blocks) of $V$ with the property that every 2-subset of $V$ appears precisely in $\lambda$ blocks. A $(v, k, \lambda)$-design is called super-simple if every pair of distinct blocks intersect at two members at most.

There are many papers on super-simple block designs, which are $(v, k, \lambda)$-designs in which any two blocks have at most two points in common; see the Handbook of Combinatorial Designs [CD] for details.

As far as we can ascertain, there is only one result (see [BCK]) on the simpler (no pun intended) problem regarding the existence of super-simple cycle systems. That is,

**Theorem 1.** There exists a super-simple twofold 4-cycle system if and only if $n \equiv 0$ or 1 (mod 4), $n \neq 4, 5, 9$.

Despite the fact that pairs of disjoint cycle systems of the same order have been constructed, combining two disjoint systems, even for 4-cycles, does not necessarily result in the twofold system being super-simple. Hence, the existence a super-simple twofold $m$-cycle system is widely open for $m \geq 5$. Note that a twofold 3-cycle system is the same as a $(v, 3, 2)$-block design.

J. Spectrum of critical sets in Latin squares: A partial Latin square $P$ of order $n$ is an $n \times n$ array with rows and columns indexed by $N = \{0, 1, 2, 3, \ldots, n-1\}$ and entries chosen from $N$ in such a way that each element of $N$ occurs at most once in each row and at most once in each column of the array.
If all the cells of the array are filled then the partial Latin square is termed a *Latin square* (see Figure 1). A partial Latin square $C$ of order $n$ is a *critical set* if $C$ is contained in exactly one Latin square of order $n$, and no proper subset of $C$ satisfies this property. Partial Latin squares $A$, $B$, and $C$ in Figure 1 are critical sets in Latin square $E$. It is easy to see that there are two Latin squares of order four containing partial Latin square $D$. Hence, $D$ is not a critical set.

The *spectrum* of critical sets in a Latin square $L$ is a set $S$ of integers such that for every $m \in S$ there is a critical set of size $m$ in $L$. It is well known that the spectrum of critical sets in Latin square $E$ in Figure 1 is $S = \{4, 5, 6\}$. The general problem of determining the spectrum of a Latin square of order $n$ is quite difficult, but for many significant classes of Latin squares it is manageable. A *back circulant* Latin square of order $n$ is a Latin square whose cell $(i, j)$ contains $i + j \pmod n$ for all $i, j \in \mathbb{Z}_n$. In [CDK] Cavenagh, Donovan and Khodkar proved that if $n$ is odd, then there exists a critical set of size $m$ in the back circulant Latin square of order $n$ for every $m \in \{n^2/4, (n^2/4)+1, (n^2/4)+2, \ldots, (n^2-n)/2\}$. The problem for $n$ even is still open.