z-SCORES
Frances Chumney, PhD
CONTENT OUTLINE

- Overview of z-Scores
- Probability & Normal Distribution
- Distribution of Sample Means
OVERVIEW OF Z-SCORES
Student A earned a score of 76 on an exam

- How many points were possible?
  - 76 out of 80? Not bad!
  - 76 out of 100? Not so great!

- How does a score of 76 compare to other students?
  - 76 the lowest score in the class?
  - Anyone earn a score higher than 76?
OVERVIEW OF z-SCORES

- **z-Score**
  
  *standardized value that specifies the exact location of an X value within a distribution by describing its distance from the mean in terms of standard deviation units*

- **Standard Deviation Unit**
  
  - Standardized value
  - 1 SD unit = value of 1 SD before standardization
OVERVIEW OF z-SCORES

σ = 3

σ = 12
**Z-Scores** describe the exact location of a score within a distribution.

- **Sign**: Whether score is above (+) or below (-) the mean.
- **Number**: Distance between score and mean in standard deviation units.

**Example**

- \( z = +1.00 \)
  - Sign: positive (+) so score is above the mean.
  - Number: 1.00 SD units from the mean.
Example

- $z = -0.50$
  - Sign: negative (-) so score is below the mean
  - Number: 0.50 SD units from the mean
FORMULA: RAW SCORE → Z-SCORE

- Transform raw score (X value) to z-Score

\[ z = \left( \frac{X - \mu}{\sigma} \right) = \left( \frac{X - M}{s} \right) \]

- Numerator = Deviation Score
- Denominator = Standard Deviation
Example

- Population A has $\mu = 5$ and $\sigma = 1$
- Find $z$-Score for $X = 3$
- $z = \left( \frac{3 - 5}{1} \right) = -2/1 = -2$
FORMULA: **RAW SCORE → Z-SCORE**

- **Example**
  - Sample B has $M = 5$ and $s = 1$
  - Find $z$-Score for $X = 5.5$
  - $z = (5.5 - 5) / 1 = 0.5 / 1 = +0.5$

\[
z = \left( \frac{X - M}{s} \right)
\]
FORMULA: RAW SCORE $\rightarrow$ Z-SCORE

- Transform $z$-Score to $X$ value (raw score)

$$X = \mu + z\sigma = M + zs$$

- 4 pieces of information:
  - $X$ = raw score
  - $\mu$ or $M$ = population/sample mean
  - $z$ = $z$-Score
  - $\sigma$ or $s$ = population/sample standard deviation
Example

- Person A from Sample Y has a z-Score of -0.75
- $\mu = 10$, $\sigma = 2$
- Find $X$ for $z$-Score = -0.75
  - $X = 10 + (-0.75)(2) = 8.5$

$X = \mu + z\sigma = M + zs$
-Scores establish relationships between score, mean, standard deviation

- Example
  - Population: $\mu = 65$ and $X = 59$ corresponds to $z = -2.00$
  - Subtract 65 from 59 and find deviation score of six points corresponds to $z$ value of -2.00
  - $(X - \mu) / z = \sigma$

- Example
  - Population: $\sigma = 4$ and $X = 33$ corresponds to $z = +1.50$
  - Multiply $\sigma$ by $z$ to find deviation score ($4 \times 1.5 = 6$)
  - Add/Subtract deviation score from $X$ to find $\mu$ ($33 - 6 = 27$)
DISTRIBUTION TRANSFORMATIONS

- **Standardized Distribution**
  
  *distribution composed of scores that have been transformed to create predetermined values for μ and σ; distributions used to make dissimilar distributions comparable*

- **Properties/Characteristics**
  
  - Same shape as original distribution – scores are renamed, but location in distribution remains same
  - Mean will always equal zero (0)
  - Standard deviation will always equal one (1)
DISTRIBUTION TRANSFORMATIONS

- **How-To**
  - Transform all $X$ values into $z$-Scores $\Rightarrow$ $z$-Score Distribution

- **Advantage**
  - Possible to compare scores or individuals from different distributions $\Rightarrow$
    Results more generalizable
    - $z$-Score distributions have equal means (0) and standard deviations (1)
STANDARDIZED DISTRIBUTIONS

- Z-Score distributions include positive and negative numbers

- Standardize to distribution with predetermined μ and σ to avoid negative values

- Procedure
  - Transform raw scores to z-scores
  - Transform z-scores into new X values with desired μ and σ values
STANDARDIZED DISTRIBUTIONS

Example

- Population distribution with $\mu = 57$ and $\sigma = 14$
- Transform distribution to have $\mu = 50$ and $\sigma = 10$
- Calculate new $X$ values for raw scores of $X = 64$ and $X = 43$
- Step 1 (of 2)
  - Transform raw scores to $z$-scores
    - $z = (X - \mu) / \sigma$
      - $z = (64 - 57) / 14 = (7 / 14) = .50$
      - $z = (43 - 57) / 14 = (-14 / 14) = -1.0$
Example (continued)

Step 2 (of 2)

- Transform to new $X$ values
  - $z = .50$ corresponds to a score $\frac{1}{2}$ of a standard deviation above the mean
  - In new distribution, $z = .50$ corresponds to score 5 points above mean ($X = 55$)
  - In new distribution, $z = -1.00$ corresponds to score 10 points below mean ($X = 40$)
PROBABILITY & NORMAL DISTRIBUTION

using the unit normal table to find proportions

Z-SCORES
PROBABILITY & NORMAL DISTRIBUTION

-2  -1  0  +1  +2  z

μ

34.13%
13.59%
2.28%
Example

\[ p(X > 80) = ? \]

- Translate into a proportion question: Out of all possible adult heights, what proportion consists of values greater than 80”?
- The set of “all possible adult heights” is the population distribution
- We are interested in all heights greater than 80”, so we shade in the area of the graph to the right of where 80” falls on the distribution
Example (continued)

- Transform $X = 80$ to a $z$-score
  \[ z = \frac{(X - \mu)}{\sigma} = \frac{(80 - 68)}{6} = \frac{12}{6} = 2.00 \]

- Express the proportion we are trying to find in terms of the $z$-score: $p(z > 2.00) = ?$

- By Figure 6.4, $p(X > 80) = p(z > +2.00) = 2.28\%$
UNIT NORMAL TABLE

Z-SCORES

-2  -1  0  +1  +2
μ

z

34.13%
13.59%
2.28%

34.13%
# UNIT NORMAL TABLE

<table>
<thead>
<tr>
<th>(A) $z$</th>
<th>(B) Proportion in body</th>
<th>(C) Proportion in tail</th>
<th>(D) Proportion between mean and $z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>.5000</td>
<td>.5000</td>
<td>.0000</td>
</tr>
<tr>
<td>0.01</td>
<td>.5040</td>
<td>.4960</td>
<td>.0040</td>
</tr>
<tr>
<td>0.02</td>
<td>.5080</td>
<td>.4920</td>
<td>.0080</td>
</tr>
<tr>
<td>0.03</td>
<td>.5120</td>
<td>.4880</td>
<td>.0120</td>
</tr>
<tr>
<td>0.21</td>
<td>.5832</td>
<td>.4168</td>
<td>.0832</td>
</tr>
<tr>
<td>0.22</td>
<td>.5871</td>
<td>.4129</td>
<td>.0871</td>
</tr>
<tr>
<td>0.23</td>
<td>.5910</td>
<td>.4090</td>
<td>.0910</td>
</tr>
<tr>
<td>0.24</td>
<td>.5948</td>
<td>.4052</td>
<td>.0948</td>
</tr>
<tr>
<td>0.25</td>
<td>.5987</td>
<td>.4013</td>
<td>.0987</td>
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<tr>
<td>0.26</td>
<td>.6026</td>
<td>.3974</td>
<td>.1026</td>
</tr>
<tr>
<td>0.27</td>
<td>.6064</td>
<td>.3936</td>
<td>.1064</td>
</tr>
<tr>
<td>0.28</td>
<td>.6103</td>
<td>.3897</td>
<td>.1103</td>
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<tr>
<td>0.29</td>
<td>.6141</td>
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<td>.1141</td>
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<td>0.30</td>
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<td>0.31</td>
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</tr>
<tr>
<td>0.32</td>
<td>.6255</td>
<td>.3745</td>
<td>.1255</td>
</tr>
<tr>
<td>0.33</td>
<td>.6293</td>
<td>.3707</td>
<td>.1293</td>
</tr>
<tr>
<td>0.34</td>
<td>.6331</td>
<td>.3669</td>
<td>.1331</td>
</tr>
</tbody>
</table>
UNIT NORMAL TABLE: GUIDELINES

- **Body** = Larger part of the distribution
- **Tail** = Smaller part of the distribution
- Distribution is symmetrical $\Rightarrow$ Proportions to right of mean are symmetrical to (read as “the same as”) those on the left side of the mean
- Proportions are always positive, even when $z$-scores are negative
- Identify proportions that correspond to $z$-scores or $z$-scores that correspond to proportions
UNIT NORMAL TABLE: COLUMN SELECTION

- Proportion in Body = Column B
UNIT NORMAL TABLE: COLUMN SELECTION

- Proportion in Tail = Column C

Z-SCORES
UNIT NORMAL TABLE: COLUMN SELECTION

- Proportion between Mean & $Z = \text{Column D}$
PROBABILITIES, PROPORTIONS, Z

- Unit Normal Table
  - Relationships between $z$-score locations and proportions in a normal distribution
  - If proportion is known, use table to identify $z$-score
  - Probability = Proportion
Example:

- Column B
  - What proportion of normal distribution corresponds to \( z \)-scores < \( z = 1.00 \)?
  - What is the probability of selecting a \( z \)-score less than \( z = 1.00 \)?

<table>
<thead>
<tr>
<th>( z )</th>
<th>(A) Proportion in Body</th>
<th>(B) Proportion in Tail</th>
<th>(C) Proportion Between Mean and ( z )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.00</td>
<td>0.8413</td>
<td>0.1587</td>
<td>0.3413</td>
</tr>
</tbody>
</table>

Answer:

\[ p(z < 1.00) = 0.8413 \text{ (or 84.13\%)} \]
Example:

- Column B
  - What proportion of a normal distribution corresponds to \( z \)-scores > \( z = -1.00 \)?
  - What is the probability of selecting a \( z \)-score greater than \( z = -1.00 \)?

**Answer:**

\[ p(z > -1.00) = .8413 \text{ (or } 84.13\%) \]
FIND PROPORTION/PROBABILITY

Example:

- Column C

  - What proportion of a normal distribution corresponds to $z$-scores $> z = 1.00$?
  - What is the probability of selecting a $z$-score value greater than $z = 1.00$?

<table>
<thead>
<tr>
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<td>0.1587</td>
<td>0.3413</td>
</tr>
</tbody>
</table>

Answer:

$p(z > 1.00) = .1587$ (or 15.87%)
Example:

- Column C
  - What proportion of a normal distribution corresponds to $z$-scores $> z = 1.00$?
  - What is the probability of selecting a $z$-score value greater than $z = 1.00$?

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Answer:

$p(z > 1.00) = .1587$ (or 15.87%)
Example:
- Column C
  - What proportion of a normal distribution corresponds to $z$-scores $< z = -1.00$?
  - What is the probability of selecting a $z$-score value less than $z = -1.00$?

Answer:
$$p(z < -1.00) = 0.1587 \text{ (or 15.87%) }$$
Example:

- Column D
  - What proportion of normal distribution corresponds to positive $z$-scores $< z = 1.00$?
  - What is the probability of selecting a positive $z$-score less than $z = 1.00$?

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<thead>
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<td>0.1587</td>
<td>0.3413</td>
</tr>
</tbody>
</table>

Answer:

$$p(0 < z < 1.00) = 0.3413 \text{ (or 34.13\%)}$$
Example:

- Column D

  - What proportion of a normal distribution corresponds to negative \( z \)-scores > \( z = -1.00 \)?
  - What is the probability of selecting a negative \( z \)-score greater than \( z = -1.00 \)?

<table>
<thead>
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<td>0.3413</td>
</tr>
</tbody>
</table>

Answer:

\[ p(0 < z < 1.00) = .3413 \text{ (or 34.13\%)} \]
Example:
- Column D
  - What proportion of a normal distribution corresponds to \( z \)-scores within 1 standard deviation of the mean?
  - What is the probability of selecting a \( z \)-score greater than \( z = -1.00 \) and less than \( z = 1.00 \)?

<table>
<thead>
<tr>
<th>( z )</th>
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</tbody>
</table>

Answer:
\[
p(-1.00 < z < 1.00) = 0.3413 + 0.3413 = 0.6826 (or 68.26\%)
\]
Example:

Column B

- What $z$-score separates the bottom 80% from the remainder of the distribution?

<table>
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<th>(B) Proportion in Tail</th>
<th>(C) Proportion Between Mean and $z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$z$</td>
<td>0.84</td>
<td>0.7995</td>
</tr>
<tr>
<td>0.84</td>
<td>0.2005</td>
<td>0.2995</td>
</tr>
</tbody>
</table>

Answer:

80% (or .8000) $\Rightarrow z = .84$
Example:

Column C

- What z-score separates the top 20% from the remainder of the distribution?

<table>
<thead>
<tr>
<th></th>
<th>(A)</th>
<th>(B)</th>
<th>(C)</th>
<th>(D)</th>
</tr>
</thead>
<tbody>
<tr>
<td>z</td>
<td>Proportion in Body</td>
<td>Proportion in Tail</td>
<td>Proportion Between Mean and z</td>
<td></td>
</tr>
<tr>
<td>0.84</td>
<td>0.7995</td>
<td>0.2005</td>
<td>0.2995</td>
<td></td>
</tr>
</tbody>
</table>

Answer:

20% (or .2000) $\Rightarrow z = .84$
Example:
- Column D
  - What \( z \)-score separates the middle 60\% from the remainder of the distribution?

<table>
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<td>0.84</td>
<td>0.7995</td>
<td>0.2005</td>
<td>0.2995</td>
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</table>

Answer:
60\% (or .6000) \( \Rightarrow \) \( z = \pm .84 \)
PROPORTION/PROBABILITY FOR X

Steps

- Convert X to \( z \)-Score
- Use Unit Normal Table to convert \( z \)-score to corresponding percentage/proportion

Example

- Assume a normal distribution with \( \mu = 100 \) and \( \sigma = 15 \)
- What is the probability of randomly selecting an individual with an IQ score less than 130?

\[
p(X < 130) = ?
\]

- Step 1: Convert \( X \) to \( z \)-Score

\[
z = \frac{X - \mu}{\sigma} = \frac{130 - 100}{15} = \frac{30}{15} = 2.00
\]
Example (continued)

- Step 2: Use Unit Normal Table to convert $z$-score to corresponding percentage/proportion

$z = 2.00$

<table>
<thead>
<tr>
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</tr>
</thead>
<tbody>
<tr>
<td>2.00</td>
<td>0.9772</td>
<td>0.0228</td>
<td>0.4772</td>
</tr>
</tbody>
</table>

Answer:

$p(X < 130) = .9772$ (or 97.72%)
Example

- Assume a normal distribution with $\mu = 58$ and $\sigma = 10$ for average speed of cars on a section of interstate highway.
- What proportion of cars traveled between 55 and 65 miles per hour?

$$p(55 < X < 65) = ?$$

- Step 1: Convert $X$ values to $z$-Scores

$$z = \frac{X - \mu}{\sigma} = \frac{55 - 58}{10} = \frac{-3}{10} = -.30$$

$$z = \frac{X - \mu}{\sigma} = \frac{65 - 58}{10} = \frac{7}{10} = .70$$
Example (continued)

- Step 2: Use Unit Normal Table to convert $z$-scores to corresponding proportions

$$z = -0.30 \quad z = 0.70$$

<table>
<thead>
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<td>0.6179</td>
<td>0.3821</td>
<td>0.1179</td>
</tr>
<tr>
<td>0.70</td>
<td>0.758</td>
<td>0.242</td>
<td>0.2580</td>
</tr>
</tbody>
</table>

Answer:

$$p(55 < X < 65) = p(-0.30 < z < +0.70) = 0.1179 + 0.2580 = 0.3759$$
Example

- Assume a normal distribution with $\mu = 58$ and $\sigma = 10$ for average speed of cars on a section of interstate highway.
- What proportion of cars traveled between 65 and 75 miles per hour?
  $$p(65 < X < 75) = ?$$
- Step 1: Convert $X$ values to $z$-Scores

$$z = \frac{X - \mu}{\sigma} = \frac{65 - 58}{10} = \frac{7}{10} = .70$$

$$z = \frac{X - \mu}{\sigma} = \frac{75 - 58}{10} = \frac{17}{10} = 1.70$$
Example (continued)

Step 2: Use Unit Normal Table to convert $z$-scores to corresponding proportions

<table>
<thead>
<tr>
<th>$z$</th>
<th>Proportion in Body</th>
<th>Proportion in Tail</th>
<th>Proportion Between Mean and $z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.70</td>
<td>0.758</td>
<td>0.242</td>
<td>0.2580</td>
</tr>
<tr>
<td>1.70</td>
<td>0.9554</td>
<td>0.0446</td>
<td>0.4554</td>
</tr>
</tbody>
</table>

Answer:

$p(65 < X < 75) = p(.70 < z < 1.70) = 0.4554 - 0.2580 = 0.1974$
DISTRIBUTION OF SAMPLE MEANS

z-scores for distributions of sample means
DISTRIBUTION OF SAMPLE MEANS

- Use of Distribution of Sample Means
  - Identify probability associated with a sample
  - Distribution = all possible $M_s$
  - Proportions = Probabilities
DISTRIBUTION OF SAMPLE MEANS

Example

- Population of SAT scores forms normal distribution with $\mu = 500$ and $\sigma = 100$. In a sample of $n = 25$ students, what is the probability that the sample mean will be greater than $M = 540$?

$$p(M > 540) = ?$$

- Central Limit Theorem describes the distribution
  - Distribution is normal because population of scores is normal
  - Distribution mean is 500 because population mean is 500
  - For $n = 25$, standard error of distribution is $\sigma_M = 20$
Example (continued)

\[ p(M > 540) = ? \]

- Step 1: Calculate standard error of the distribution

\[
\sigma_M = \frac{\sigma}{\sqrt{n}} = \frac{100}{\sqrt{25}} = \frac{100}{5} = 20
\]

- Step 2: Calculate corresponding \( z \)-score

\[
z = \frac{(M - \mu)}{\sigma_M} = \frac{(540 - 500)}{20} = \frac{40}{20} = 2
\]
Example (continued)

\[ p(M > 540) = ? \]

- Step 3: Unit normal table to find correct value of \( p \) corresponding to shaded area for \( z \).

\[ p(M > 540) = 0.0228 \]
Z-SCORE FOR SAMPLE MEANS

- Where a sample is located relative to all other possible samples

- Formula

\[ z = \frac{(M - \mu)}{\sigma_M} \]

- Applications
  - Probabilities associated with specific means
  - Predict kinds of samples obtainable from a population
Example

- Predict kinds of samples obtainable from a population
- The distribution of SAT scores is normally distributed with a mean of \( \mu = 500 \) and a standard deviation of \( \sigma = 100 \). Determine what kind of sample mean is likely to be obtained as the average SAT score for a random sample of \( n = 25 \) students 80% of the time.
Example (continued)

- Determine what kind of sample mean is likely to be obtained as the average SAT score for a random sample of \( n = 25 \) students 80% of the time.
Example (continued)

- Determine what kind of sample mean is likely to be obtained as the average SAT score for a random sample of $n = 25$ students 80% of the time.
  - $z = -1.28$ and $1.28$
  - Last Step: Calculate mean values

\[
M = \mu + z \sigma_M = 500 + (-1.28 \times 20) = 500 - 25.6 = 474.4
\]
\[
M = \mu + z \sigma_M = 500 + (1.28 \times 20) = 500 + 25.6 = 525.6
\]

- 80% of sample means fall between 474.4 and 525.6