z-SCORES

Frances Chumney, PhD

CONTENT OUTLINE

- Overview of *z*-Scores
- Probability & Normal Distribution
- Distribution of Sample Means

OVERVIEW OF Z-SCORES

OVERVIEW OF z-SCORES

- Student A earned a score of 76 on an exam
  - How many points were possible?
    - 76 out of 80? Not bad!
    - 76 out of 100? Not so great!
  - How does a score of 76 compare to other students?
    - 76 the lowest score in the class?
    - Anyone earn a score higher than 76?

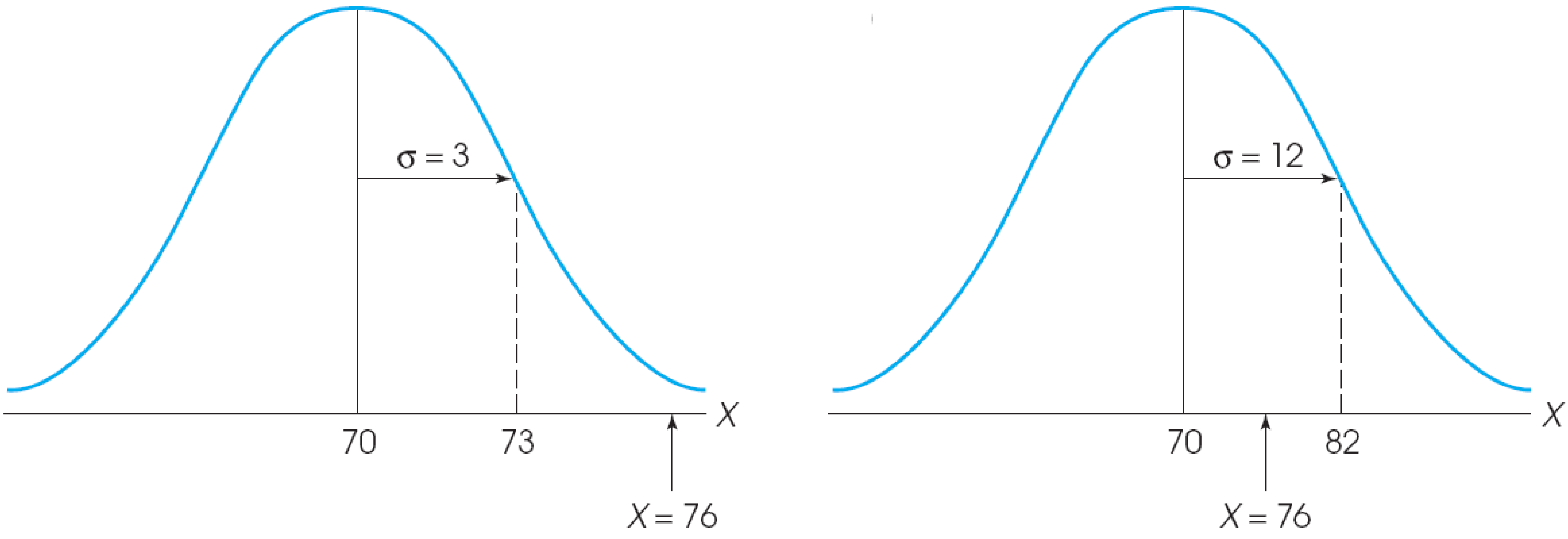
OVERVIEW OF z-SCORES

- **Z-Score**

  standardized value that specifies the exact location of an X value within a distribution by describing its distance from the mean in terms of standard deviation units

- Standard Deviation Unit
  - Standardized value
  - 1 *SD* unit = value of 1 *SD* before standardization

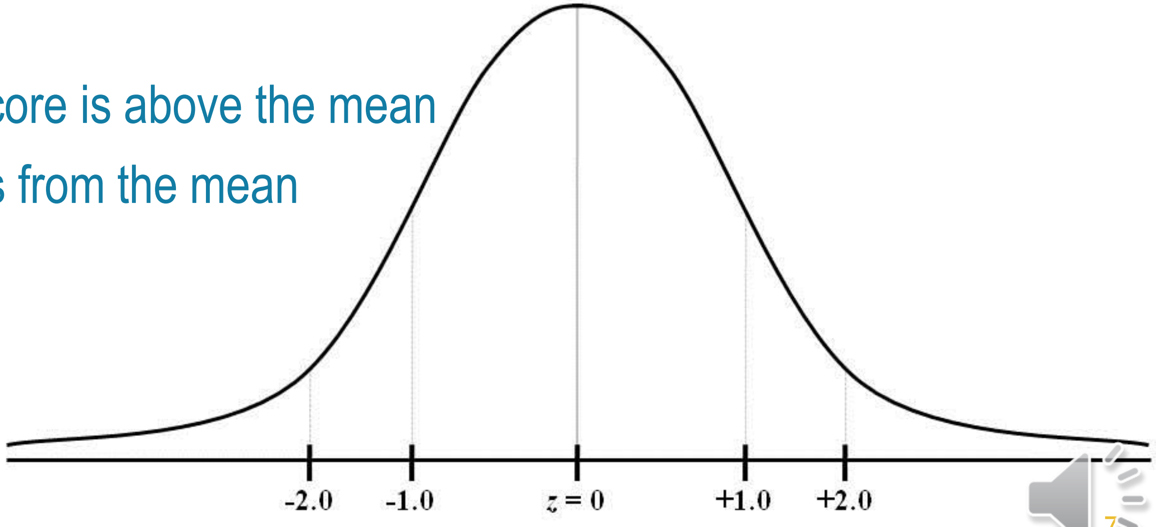
OVERVIEW OF z-SCORES



SCORE LOCATION

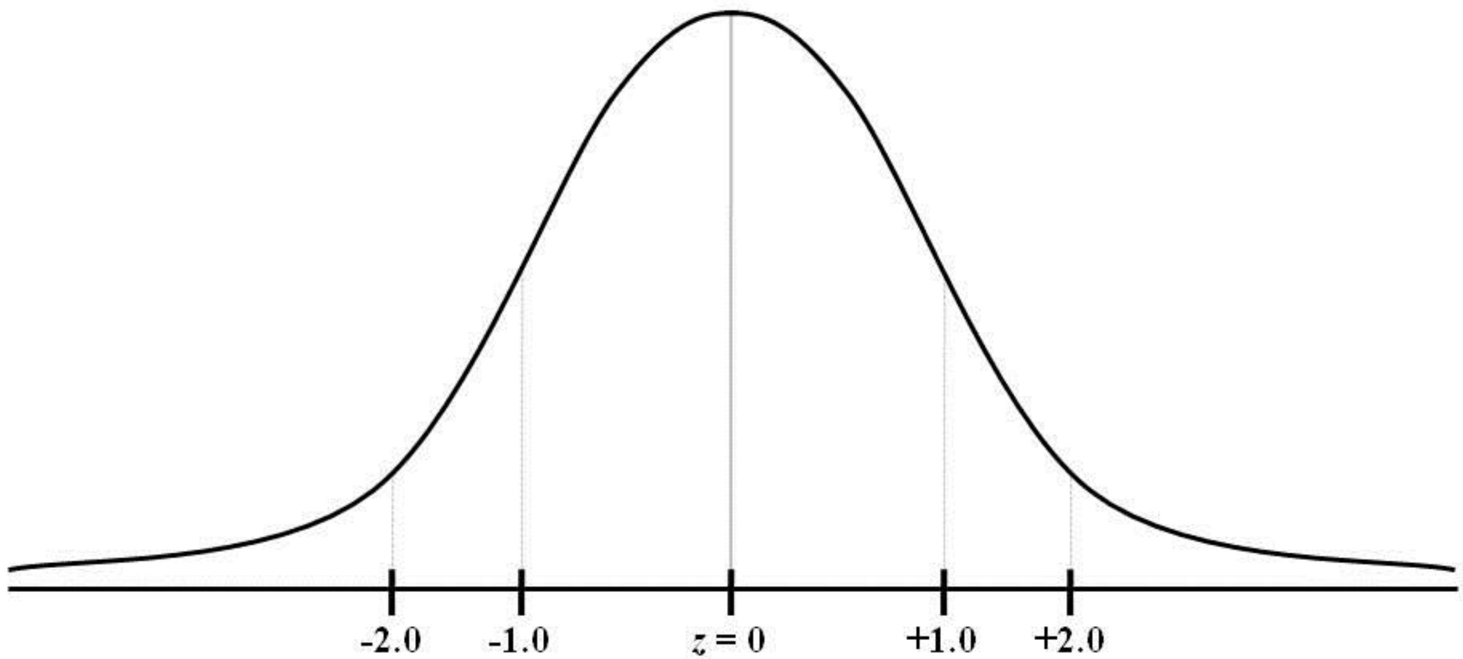
- *Z*-Scores describe the exact location of a score within a distribution
  - Sign: Whether score is above (+) or below (-) the mean
  - Number: Distance between score and mean in standard deviation units

- Example
  - z = +1.00
    - Sign: positive (+) so score is above the mean
    - Number: 1.00 *SD* units from the mean



SCORE LOCATION

- Example
  - z = - .50
    - Sign: negative (-) so score is below the mean
    - Number: .50 *SD* units from the mean



FORMULA: RAW SCORE → z-SCORE

- Transform raw score (X value) to z-Score

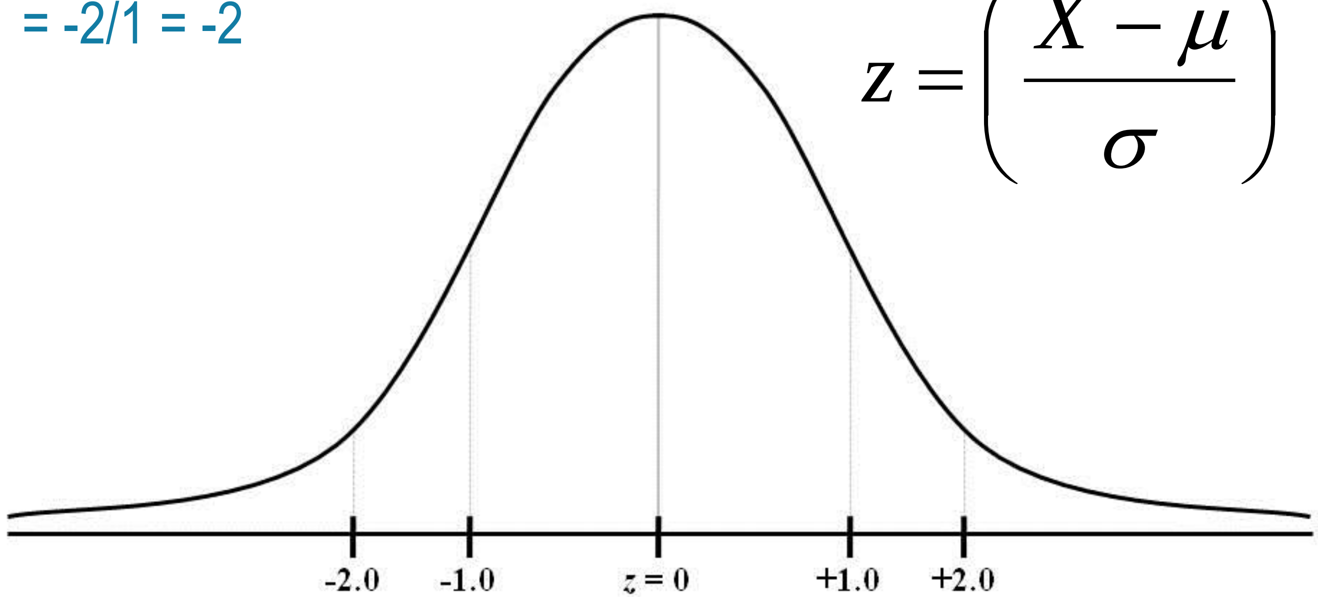
$$z = \left(\frac{X - \mu}{\sigma}\right) = \left(\frac{X - M}{s}\right)$$

  - Numerator = Deviation Score
  - Denominator = Standard Deviation

FORMULA: RAW SCORE → z-SCORE

- Example
  - Population A has $\mu = 5$ and $\sigma = 1$
  - Find z-Score for $X = 3$
  - $z = (3-5) / 1 = -2/1 = -2$

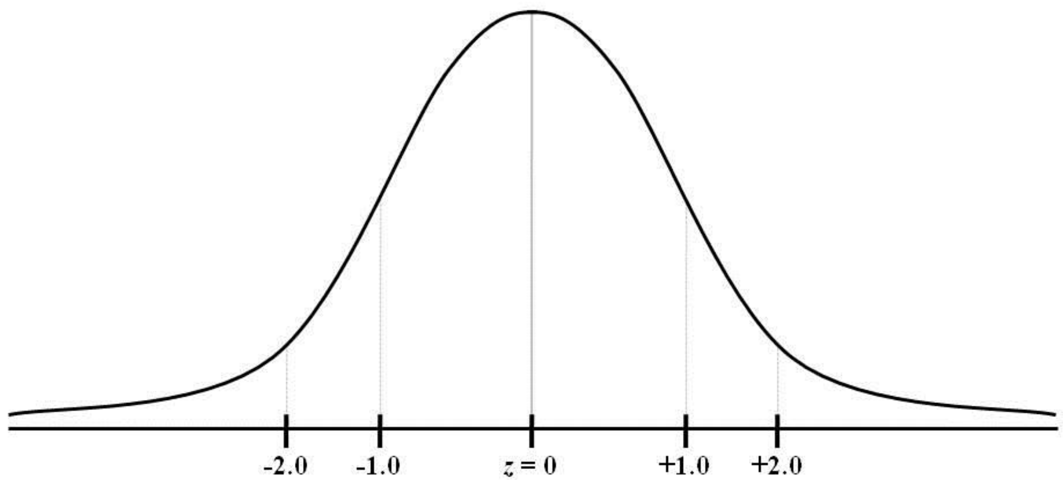
$$z = \left(\frac{X - \mu}{\sigma} \right)$$



FORMULA: RAW SCORE → z-SCORE

- Example
  - Sample B has M = 5 and s = 1
  - Find z-Score for X = 5.5
  - z = (5.5-5) / 1 = .5/1 = +.5

$$z = \left(\frac{X - M}{s} \right)$$



FORMULA: RAW SCORE → z-SCORE

- Transform z-Score to X value (raw score)

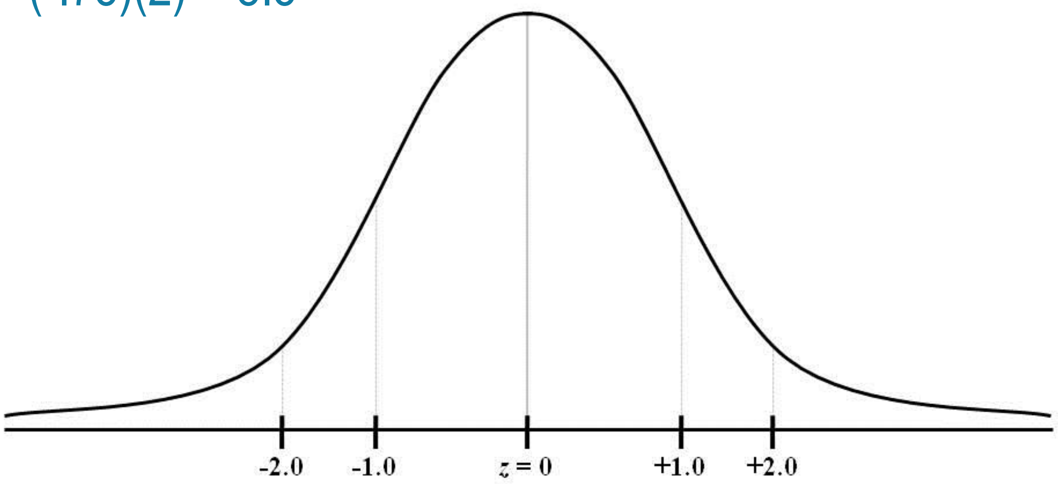
$$X = \mu + z\sigma = M + zs$$

  - 4 pieces of information:
    - X = raw score
    - μ or M = population/sample mean
    - z = z-Score
    - σ or s = population/sample standard deviation

FORMULA: RAW SCORE → *Z*-SCORE

- Example
  - Person A from Sample Y has a *z*-Score of -.75
  - $\mu = 10, \sigma = 2$
  - Find X for z-Score = -.75
    - $X = 10 + (-.75)(2) = 8.5$

$$X = \mu + z\sigma = M + zs$$



RELATIONSHIPS

- z-Scores establish relationships between score, mean, standard deviation
  - Example
    - Population: $\mu = 65$ and $X = 59$ corresponds to $z = -2.00$
    - Subtract 65 from 59 and find deviation score of six points corresponds to z value of -2.00
    - $(X - \mu) / z = \sigma$
  - Example
    - Population: $\sigma = 4$ and $X = 33$ corresponds to $z = +1.50$
    - Multiply σ by z to find deviation score (4 * 1.5 = 6)
    - Add/Subtract deviation score from X to find μ (33 – 6 = 27)

DISTRIBUTION TRANSFORMATIONS

- **Standardized Distribution**

  distribution composed of scores that have been transformed to create predetermined values for μ and σ; distributions used to make dissimilar distributions comparable

- Properties/Characteristics
  - Same shape as original distribution – scores are renamed, but location in distribution remains same
  - Mean will always equal zero (0)
  - Standard deviation will always equal one (1)

DISTRIBUTION TRANSFORMATIONS

- How-To
  - Transform all X values into z-Scores ⇨ z-Score Distribution

- Advantage
  - Possible to compare scores or individuals from different distributions ⇨ Results more generalizable
    - z-Score distributions have equal means (0) and standard deviations (1)

STANDARDIZED DISTRIBUTIONS

- z-Score distributions include positive and negative numbers
- Standardize to distribution with predetermined μ and σ to avoid negative values
- Procedure
  - Transform raw scores to z-scores
  - Transform z-scores into new X values with desired μ and σ values

STANDARDIZED DISTRIBUTIONS

- Example
  - Population distribution with μ = 57 and σ = 14
  - Transform distribution to have μ = 50 and σ = 10
  - Calculate new X values for raw scores of X = 64 and X = 43
  - Step 1 (of 2)
    - Transform raw scores to z-scores
      - $z = (X - \mu) / \sigma$
        - ✓ z = (64 – 57) / 14 = (7 / 14) = .50
        - ✓ z = (43 – 57) / 14 = (-14 / 14) = -1.0

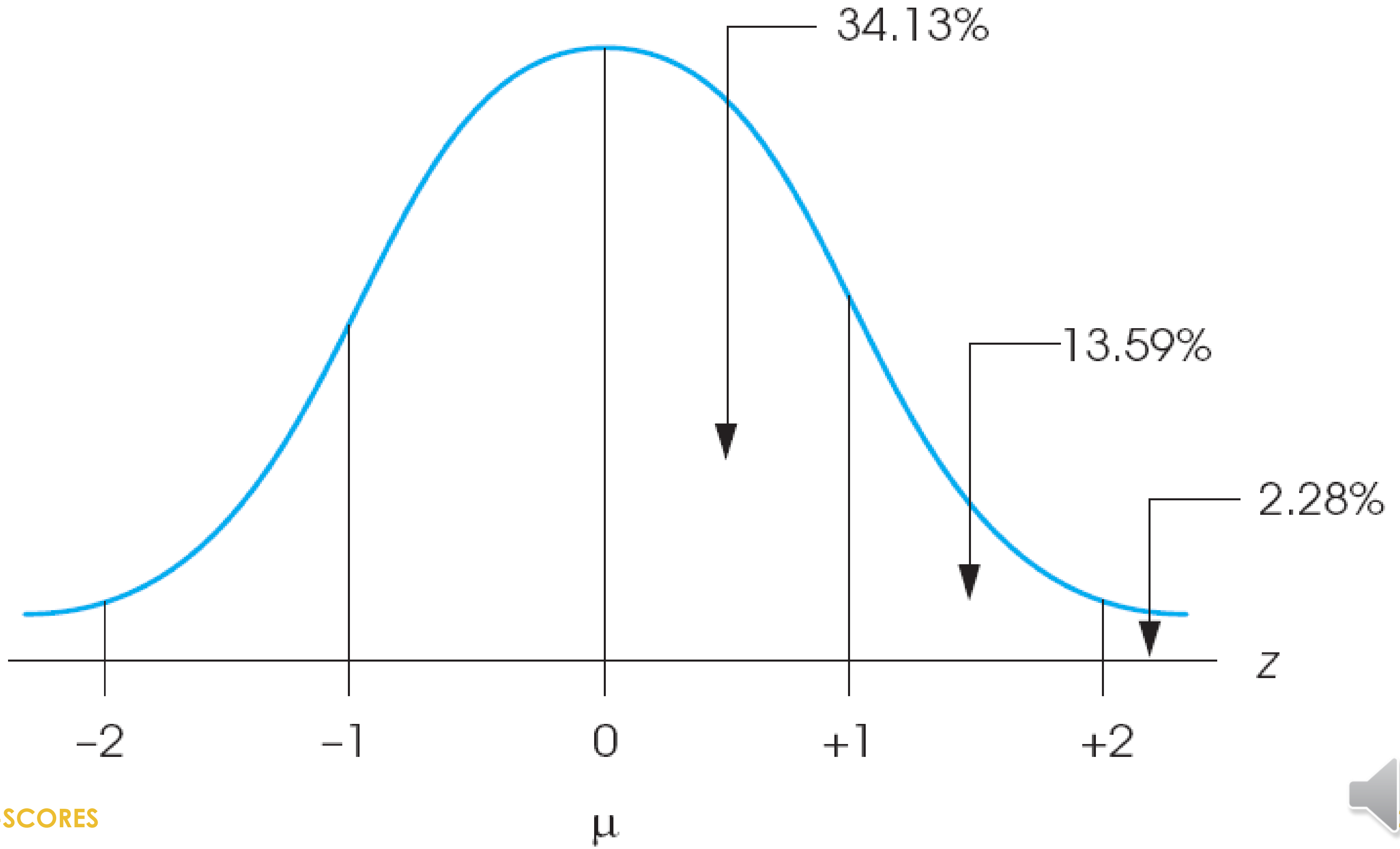
STANDARDIZED DISTRIBUTIONS

- Example (continued)
  - Step 2 (of 2)
    - Transform to new X values
      - z = .50 corresponds to a score ½ of a standard deviation above the mean
      - In new distribution, z = .50 corresponds to score 5 points above mean (X = 55)
      - In new distribution, z = -1.00 corresponds to score 10 points below mean (X = 40)

using the unit normal table to find proportions

PROBABILITY & NORMAL DISTRIBUTION

PROBABILITY & NORMAL DISTRIBUTION



PROBABILITY & NORMAL DISTRIBUTION

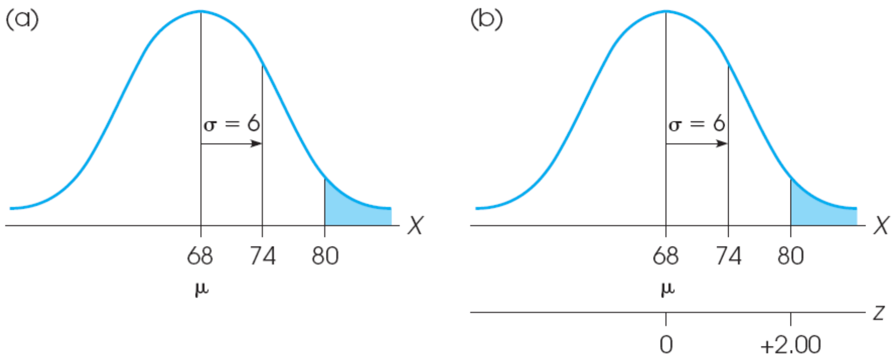
- Example
  - $p(X > 80) = ?$
    - Translate into a proportion question: Out of all possible adult heights, what proportion consists of values greater than 80"?
    - The set of "all possible adult heights" is the population distribution
    - We are interested in all heights greater than 80", so we shade in the area of the graph to the right of where 80" falls on the distribution

PROBABILITY & NORMAL DISTRIBUTION

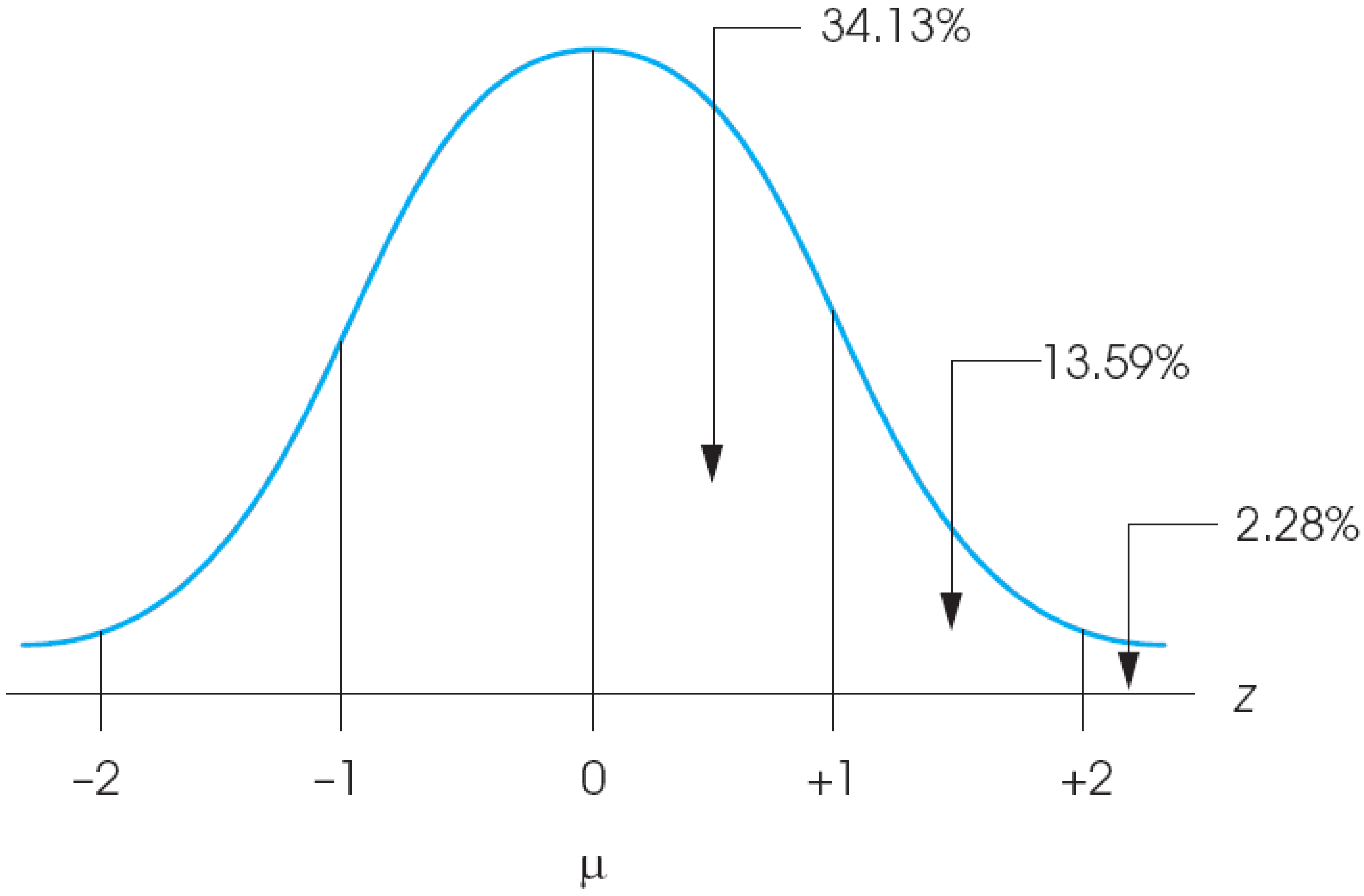
- Example (continued)
  - Transform $X = 80$ to a z-score

$$z = (X - \mu) / \sigma = (80 - 68) / 6 = 12 / 6 = 2.00$$

  - Express the proportion we are trying to find in terms of the z-score: $p(z > 2.00) = ?$
  - By Figure 6.4, $p(X > 80) = p(z > +2.00) = 2.28\%$

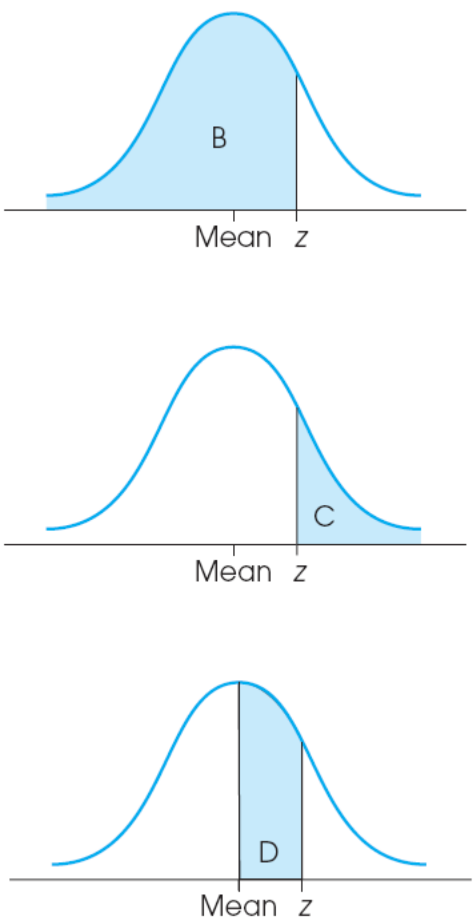


UNIT NORMAL TABLE



UNIT NORMAL TABLE

(A) z	(B) Proportion in body	(C) Proportion in tail	(D) Proportion between mean and z
0.00	.5000	.5000	.0000
0.01	.5040	.4960	.0040
0.02	.5080	.4920	.0080
0.03	.5120	.4880	.0120
0.21	.5832	.4168	.0832
0.22	.5871	.4129	.0871
0.23	.5910	.4090	.0910
0.24	.5948	.4052	.0948
0.25	.5987	.4013	.0987
0.26	.6026	.3974	.1026
0.27	.6064	.3936	.1064
0.28	.6103	.3897	.1103
0.29	.6141	.3859	.1141
0.30	.6179	.3821	.1179
0.31	.6217	.3783	.1217
0.32	.6255	.3745	.1255
0.33	.6293	.3707	.1293
0.34	.6331	.3669	.1331

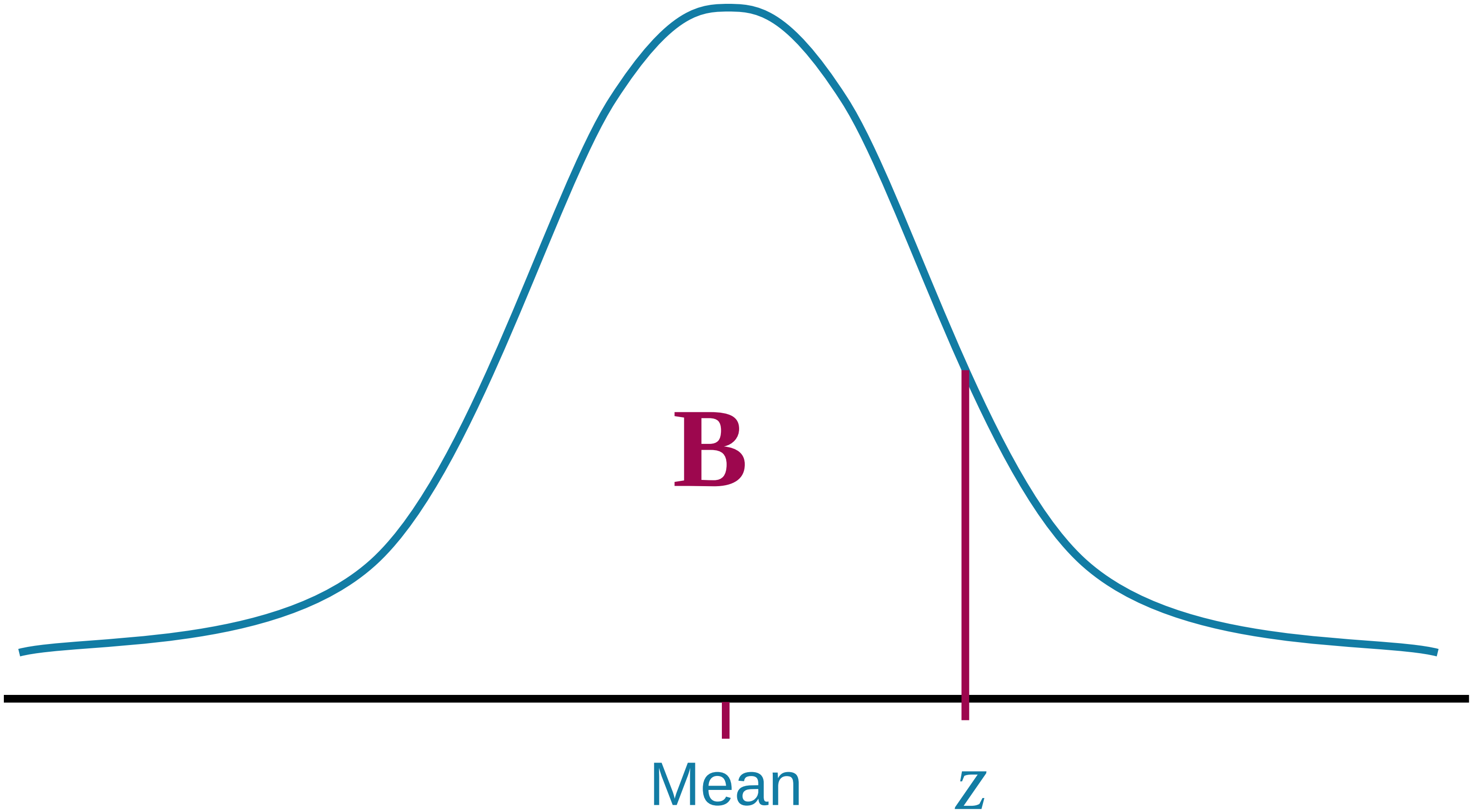


UNIT NORMAL TABLE: GUIDELINES

- Body = Larger part of the distribution
- Tail = Smaller part of the distribution
- Distribution is symmetrical ⇨ Proportions to right of mean are symmetrical to (read as "the same as") those on the left side of the mean
- Proportions are always positive, even when z-scores are negative
- Identify proportions that correspond to z-scores or z-scores that correspond to proportions

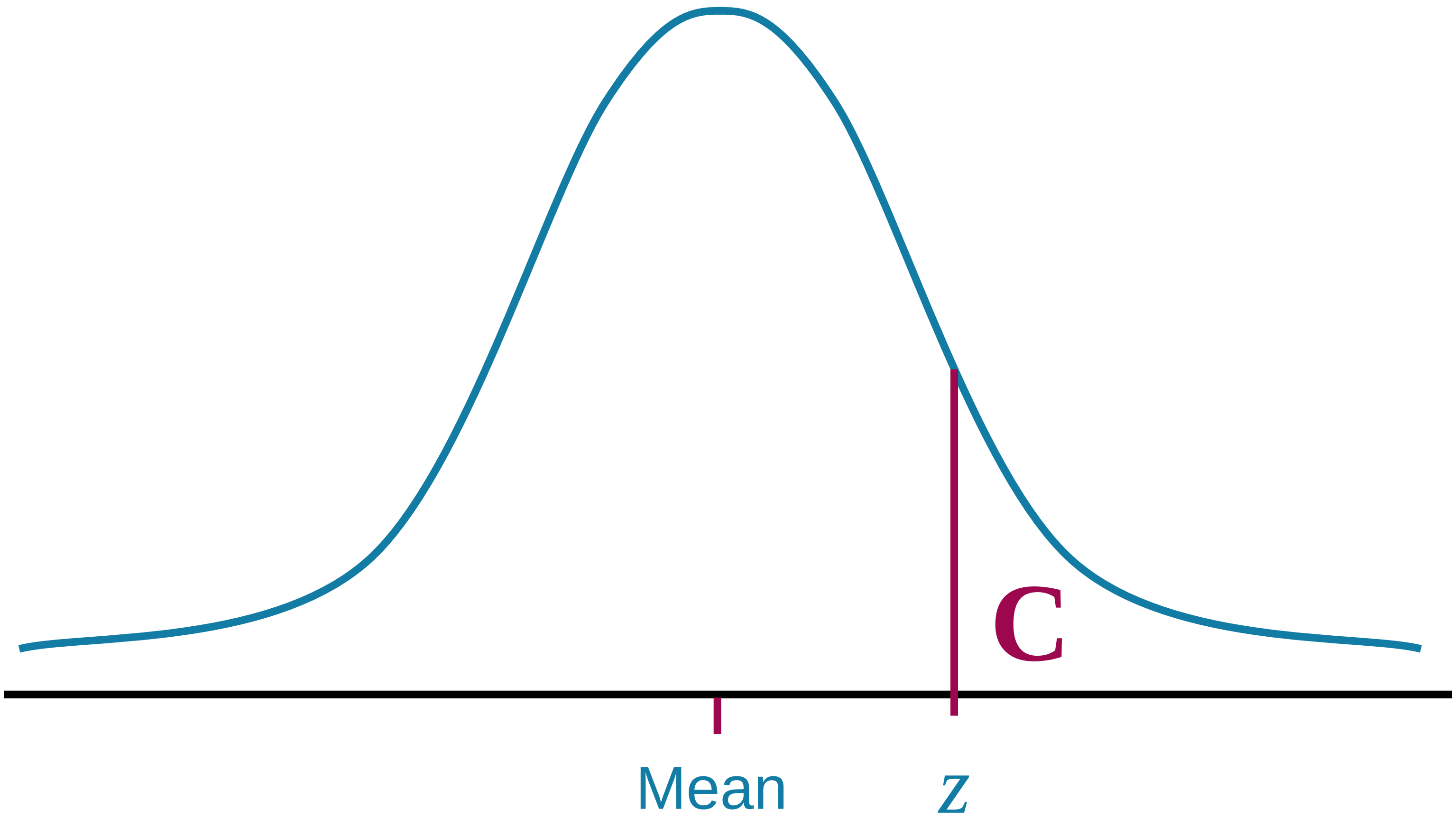
UNIT NORMAL TABLE: COLUMN SELECTION

- Proportion in Body = Column B



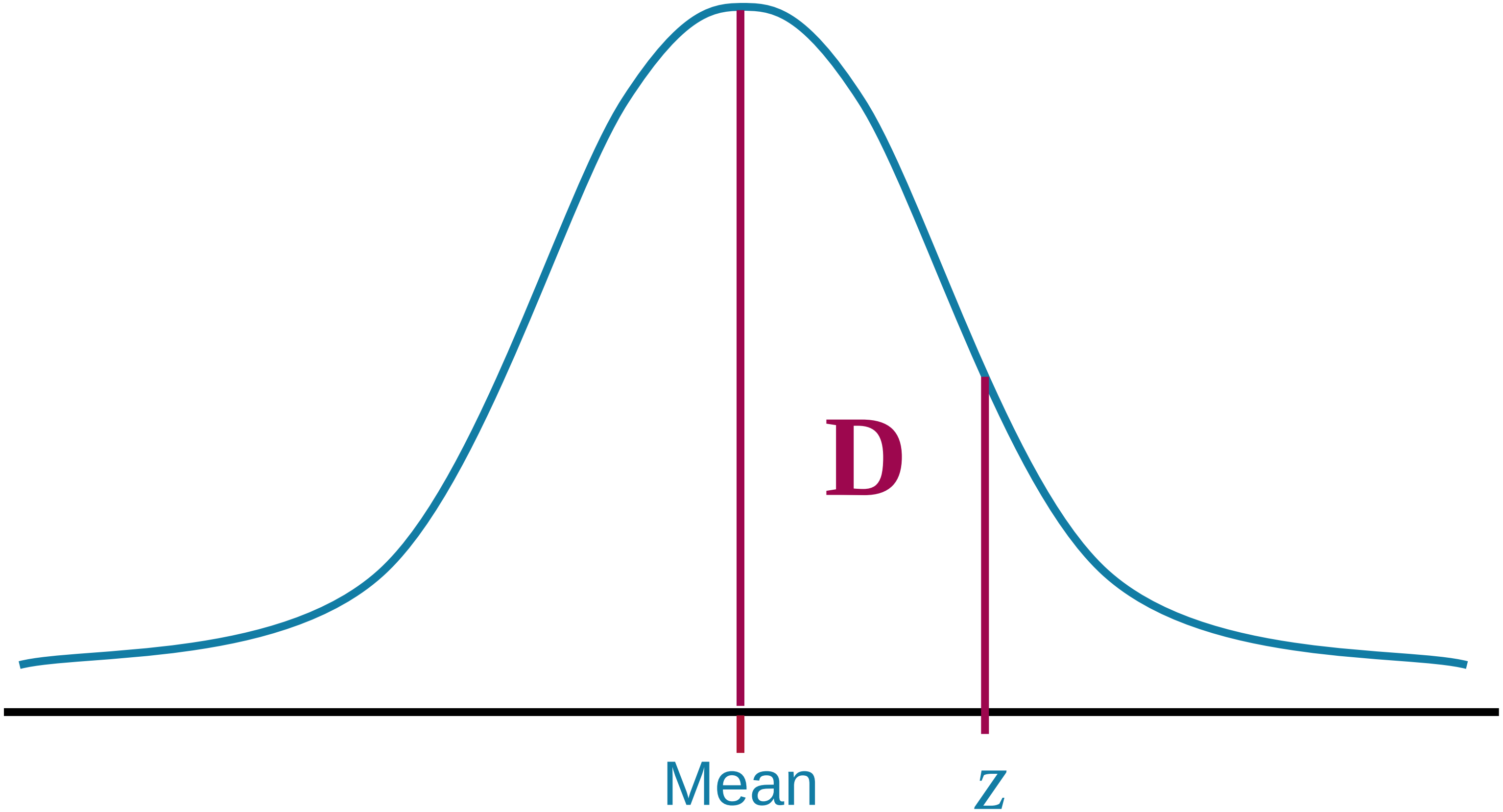
UNIT NORMAL TABLE: COLUMN SELECTION

- Proportion in Tail = Column C



UNIT NORMAL TABLE: COLUMN SELECTION

- Proportion between Mean & z = Column D

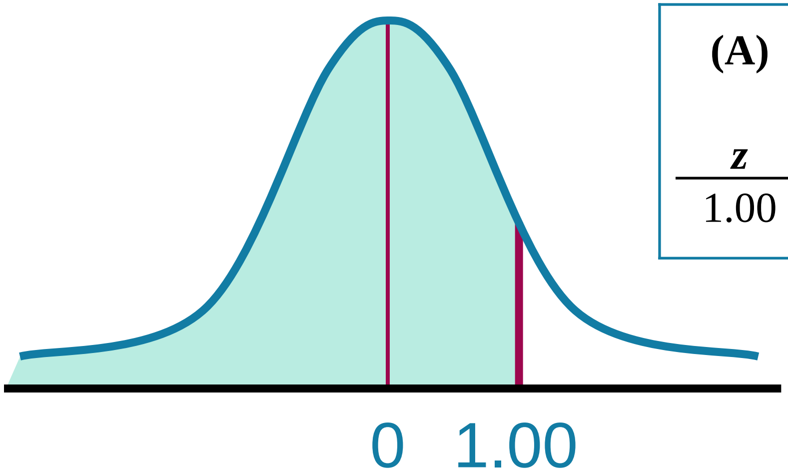


PROBABILITIES, PROPORTIONS, Z

- Unit Normal Table
  - Relationships between z-score locations and proportions in a normal distribution
  - If proportion is known, use table to identify z-score
  - Probability = Proportion

FIND PROPORTION/PROBABILITY

- Example:
  - Column B
    - What proportion of normal distribution corresponds to z-scores < z = 1.00?
    - What is the probability of selecting a z-score less than z = 1.00?



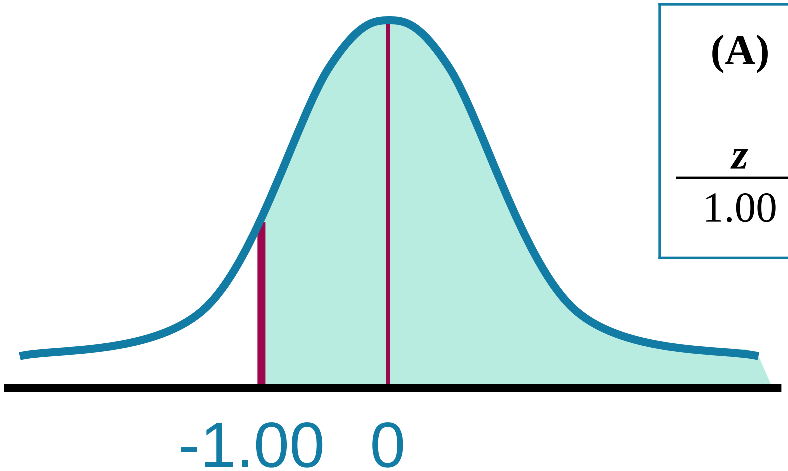
(A) z	(B) Proportion in Body	(C) Proportion in Tail	(D) Proportion Between Mean and z
1.00	**0.8413**	0.1587	0.3413

- Answer:

  $p(z < 1.00) = .8413$ (or 84.13%)

FIND PROPORTION/PROBABILITY

- Example:
  - Column B
    - What proportion of a normal distribution corresponds to z-scores > z = -1.00?
    - What is the probability of selecting a z-score greater than z = -1.00?



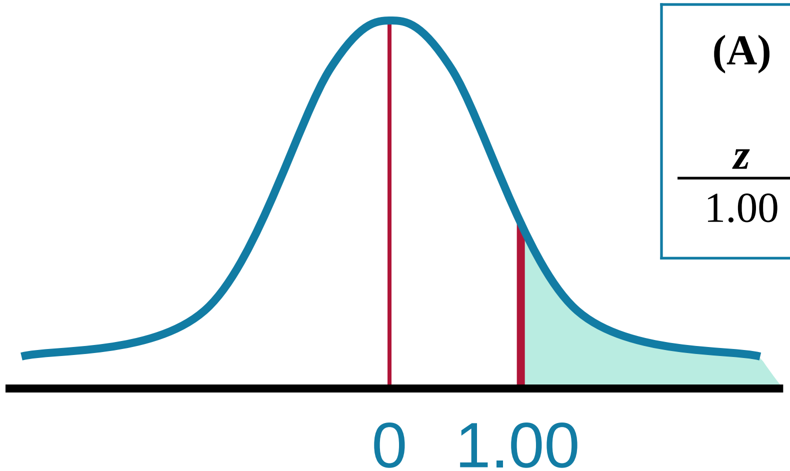
(A) z	(B) Proportion in Body	(C) Proportion in Tail	(D) Proportion Between Mean and z
1.00	**0.8413**	0.1587	0.3413

- Answer:

$p(z > -1.00) = .8413$ (or 84.13%)

FIND PROPORTION/PROBABILITY

- Example:
  - Column C
    - What proportion of a normal distribution corresponds to z-scores > z = 1.00?
    - What is the probability of selecting a z-score value greater than z = 1.00?



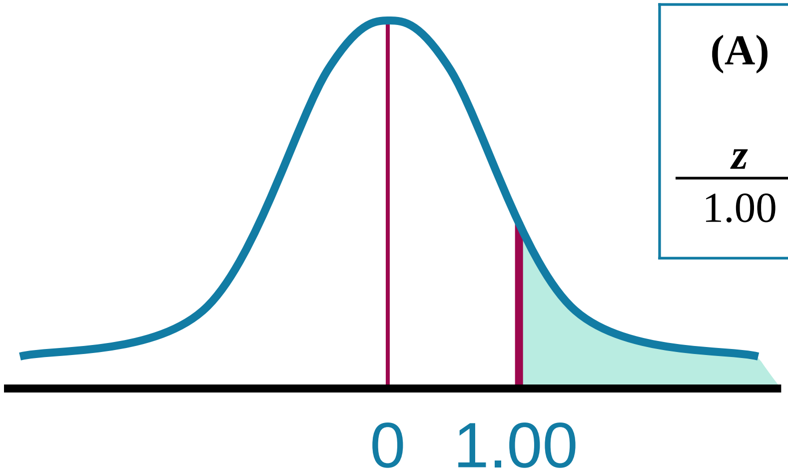
(A) z	(B) Proportion in Body	(C) Proportion in Tail	(D) Proportion Between Mean and z
1.00	0.8413	**0.1587**	0.3413

- Answer:

$p(z > 1.00) = .1587$ (or 15.87%)

FIND PROPORTION/PROBABILITY

- Example:
  - Column C
    - What proportion of a normal distribution corresponds to z-scores > z = 1.00?
    - What is the probability of selecting a z-score value greater than z = 1.00?



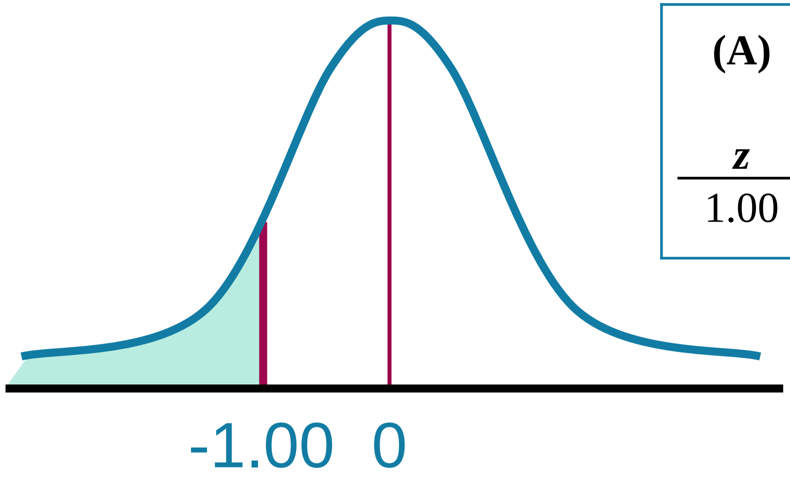
(A) z	(B) Proportion in Body	(C) Proportion in Tail	(D) Proportion Between Mean and z
1.00	0.8413	**0.1587**	0.3413

- Answer:

$p(z > 1.00) = .1587$ (or 15.87%)

FIND PROPORTION/PROBABILITY

- Example:
  - Column C
    - What proportion of a normal distribution corresponds to z-scores < z = -1.00?
    - What is the probability of selecting a z-score value less than z = -1.00?



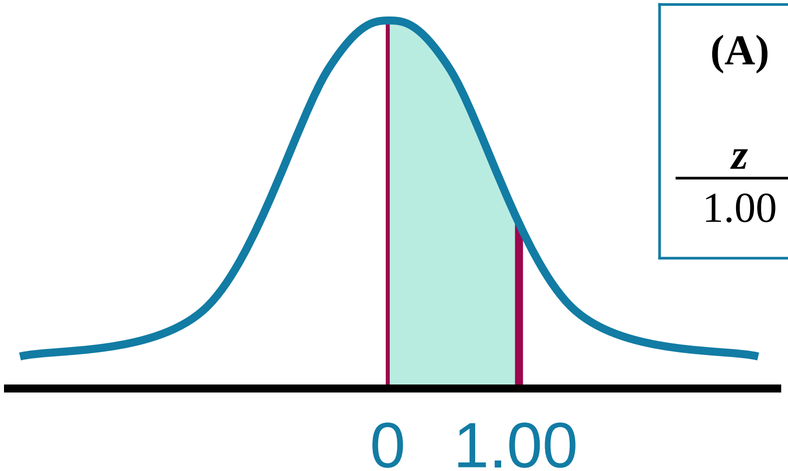
(A) z	(B) Proportion in Body	(C) Proportion in Tail	(D) Proportion Between Mean and z
1.00	0.8413	**0.1587**	0.3413

- Answer:

$p(z < -1.00) = .1587$ (or 15.87%)

FIND PROPORTION/PROBABILITY

- Example:
  - Column D
    - What proportion of normal distribution corresponds to positive z-scores < z = 1.00?
    - What is the probability of selecting a positive z-score less than z = 1.00?



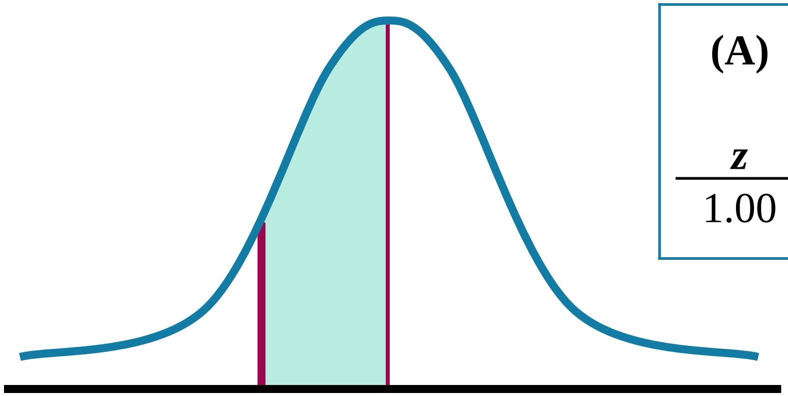
(A) z	(B) Proportion in Body	(C) Proportion in Tail	(D) Proportion Between Mean and z
1.00	0.8413	0.1587	**0.3413**

- Answer:

  $p(0 < z < 1.00) = .3413$ (or 34.13%)

FIND PROPORTION/PROBABILITY

- Example:
  - Column D
    - What proportion of a normal distribution corresponds to negative z-scores > z = -1.00?
    - What is the probability of selecting a negative z-score greater than z = -1.00?



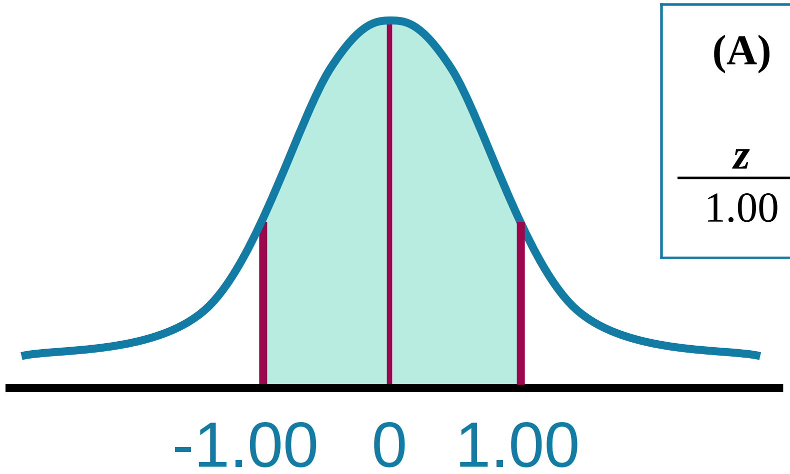
(A) z	(B) Proportion in Body	(C) Proportion in Tail	(D) Proportion Between Mean and z
1.00	0.8413	0.1587	**0.3413**

- Answer:

  $p(0 < z < 1.00) = .3413$ (or 34.13%)

FIND PROPORTION/PROBABILITY

- Example:
  - Column D
    - What proportion of a normal distribution corresponds to *z*-scores within 1 standard deviation of the mean?
    - What is the probability of selecting a *z*-score greater than *z* = -1.00 and less than *z* = 1.00 ?



(A) *z*	(B) Proportion in Body	(C) Proportion in Tail	(D) Proportion Between Mean and *z*
1.00	0.8413	0.1587	**0.3413**

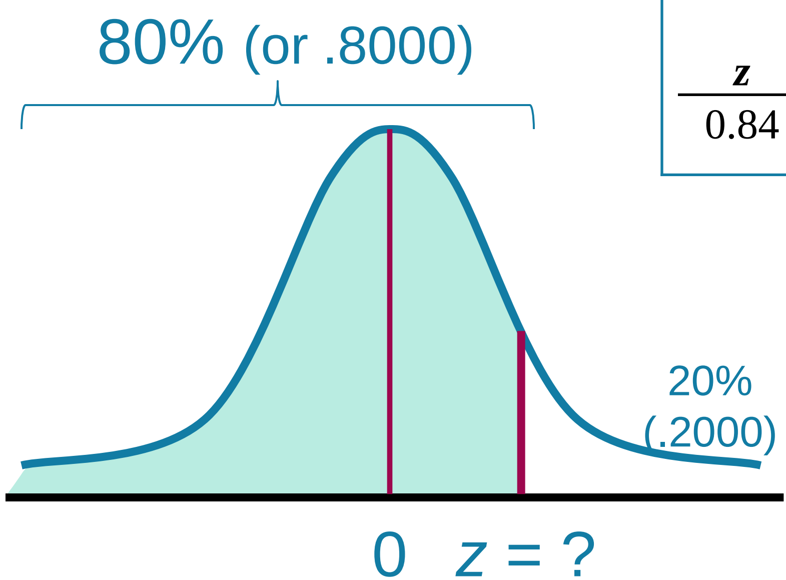
- Answer:

  .3413 + .3413 = .6826

  $p(-1.00 < z < 1.00) = .6826$ (or 68.26%)

FIND Z-SCORE

- Example:
  - Column B
    - What z-score separates the bottom 80% from the remainder of the distribution?



(A) z	(B) Proportion in Body	(C) Proportion in Tail	(D) Proportion Between Mean and z
0.84	**0.7995**	0.2005	0.2995

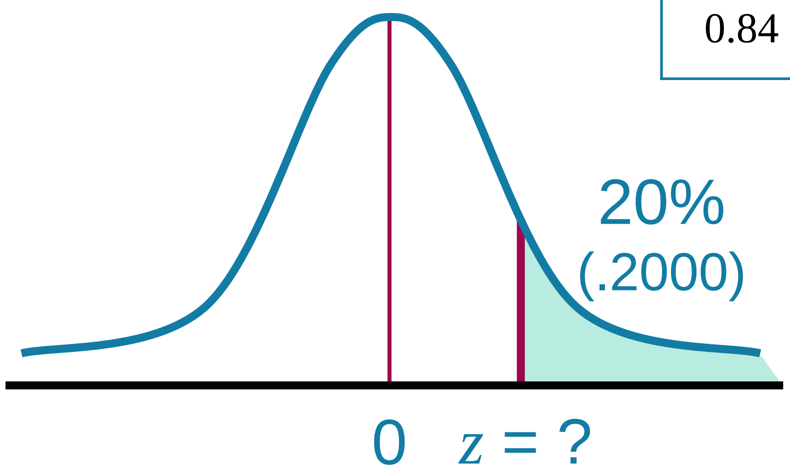
- Answer:

80% (or .8000) ⇨ z = .84

FIND Z-SCORE

- Example:
  - Column C
    - What z-score separates the top 20% from the remainder of the distribution?

(A) z	(B) Proportion in Body	(C) Proportion in Tail	(D) Proportion Between Mean and z
0.84	0.7995	**0.2005**	0.2995

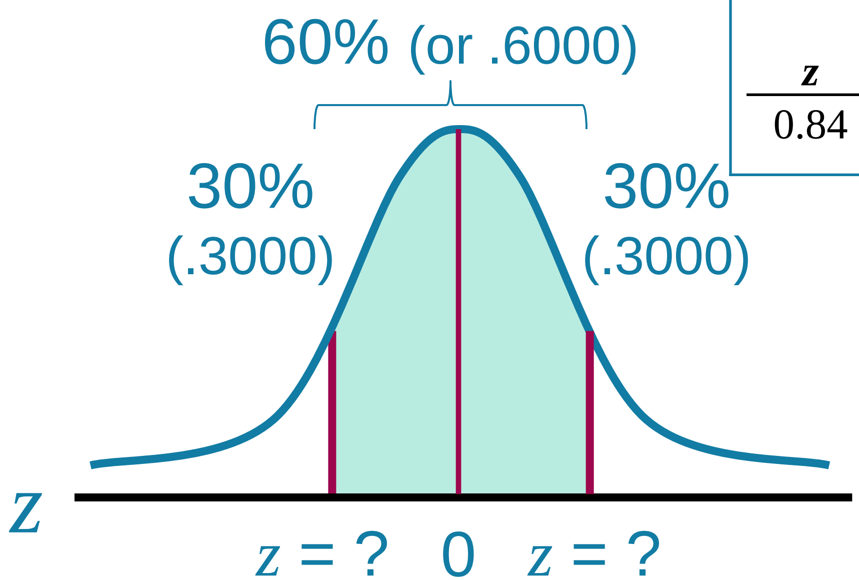


- Answer:

20% (or .2000) ⇨ z = .84

FIND Z-SCORE

- Example:
  - Column D
    - What *z*-score separates the middle 60% from the remainder of the distribution?



(A) z	(B) Proportion in Body	(C) Proportion in Tail	(D) Proportion Between Mean and z
0.84	0.7995	0.2005	**0.2995**

- Answer:

  60% (or .6000) ⇨ $z = \pm .84$

PROPORTION/PROBABILITY FOR X

- Steps
  - Convert X to z-Score
  - Use Unit Normal Table to convert z-score to corresponding percentage/proportion
- Example
  - Assume a normal distribution with $\mu = 100$ and $\sigma = 15$
  - What is the probability of randomly selecting an individual with an IQ score less than 130?

$$p(X < 130) = ?$$

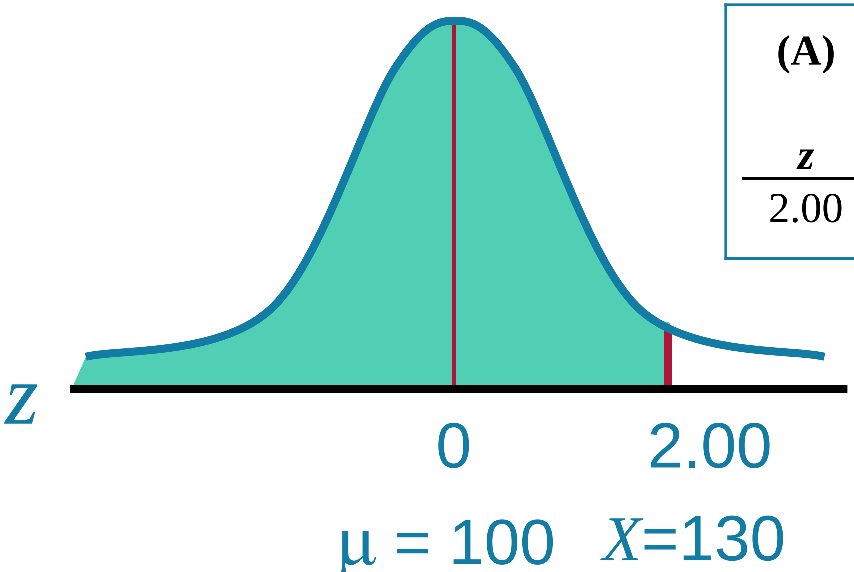
  - Step 1: Convert X to z-Score

$$z = \frac{X - \mu}{\sigma} = \frac{130 - 100}{15} = \frac{30}{15} = 2.00$$

PROPORTION/PROBABILITY FOR X

- Example (continued)
  - Step 2: Use Unit Normal Table to convert z-score to corresponding percentage/proportion

$z = 2.00$



(A) z	(B) Proportion in Body	(C) Proportion in Tail	(D) Proportion Between Mean and z
2.00	**0.9772**	**0.0228**	0.4772

- Answer:

  $p(X < 130) = .9772$ (or 97.72%)

PROPORTION/PROBABILITY FOR X

- Example
  - Assume a normal distribution with $\mu = 58$ and $\sigma = 10$ for average speed of cars on a section of interstate highway
  - What proportion of cars traveled between 55 and 65 miles per hour?

$$p(55 < X < 65) = ?$$

  - Step 1: Convert X values to z-Scores

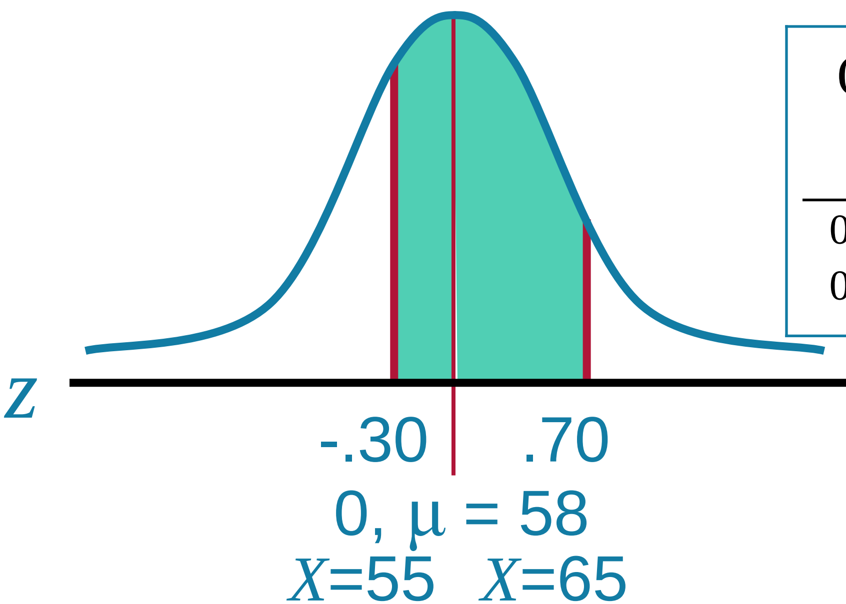
$$z = \frac{X - \mu}{\sigma} = \frac{55 - 58}{10} = \frac{-3}{10} = -.30$$

$$z = \frac{X - \mu}{\sigma} = \frac{65 - 58}{10} = \frac{7}{10} = .70$$

PROPORTION/PROBABILITY FOR X

- Example (continued)
  - Step 2: Use Unit Normal Table to convert z-scores to corresponding proportions

z = -.30 z = .70



(A) z	(B) Proportion in Body	(C) Proportion in Tail	(D) Proportion Between Mean and z
0.30	0.6179	0.3821	**0.1179**
0.70	0.758	0.242	**0.2580**

- Answer:

$p(55 < X < 65) = p(-.30 < z < +.70) = 0.1179$
$+ 0.2580 = 0.3759$

PROPORTION/PROBABILITY FOR X

- Example
  - Assume a normal distribution with $\mu = 58$ and $\sigma = 10$ for average speed of cars on a section of interstate highway
  - What proportion of cars traveled between 65 and 75 miles per hour?

$$p(65 < X < 75) = ?$$

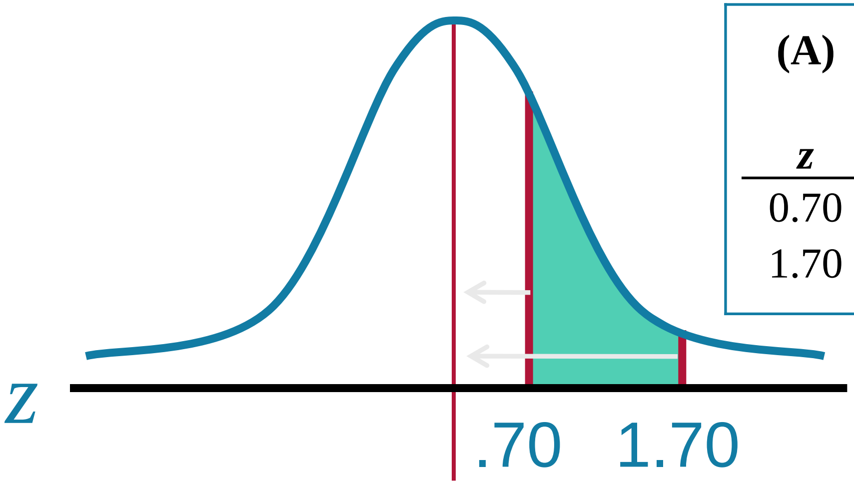
  - Step 1: Convert X values to z-Scores

$$z = \frac{X - \mu}{\sigma} = \frac{65 - 58}{10} = \frac{7}{10} = .70 \qquad z = \frac{X - \mu}{\sigma} = \frac{75 - 58}{10} = \frac{17}{10} = 1.70$$

PROPORTION/PROBABILITY FOR X

- Example (continued)
  - Step 2: Use Unit Normal Table to convert z-scores to corresponding proportions

$z = .70$ $z = 1.70$



(A) z	(B) Proportion in Body	(C) Proportion in Tail	(D) Proportion Between Mean and z
0.70	0.758	0.242	**0.2580**
1.70	0.9554	0.0446	**0.4554**

- Answer:

$p(65 < X < 75) = p(.70 < z < 1.70) = 0.4554 - 0.2580 = 0.1974$

z-scores for distributions of sample means

DISTRIBUTION OF SAMPLE MEANS

DISTRIBUTION OF SAMPLE MEANS

- Use of Distribution of Sample Means
  - Identify probability associated with a sample
  - Distribution = all possible M_s
  - Proportions = Probabilities

DISTRIBUTION OF SAMPLE MEANS

- Example
  - Population of SAT scores forms normal distribution with $\mu = 500$ and $\sigma = 100$. In a sample of $n = 25$ students, what is the probability that the sample mean will be greater than $M = 540$?

$$p(M > 540) = ?$$

  - Central Limit Theorem describes the distribution
    - Distribution is normal because population of scores is normal
    - Distribution mean is 500 because population mean is 500
    - For $n = 25$, standard error of distribution is $\sigma_M = 20$

DISTRIBUTION OF SAMPLE MEANS

- Example (continued)

$p(M > 540) = ?$

- Step 1: Calculate standard error of the distribution

$$\sigma_M = \frac{\sigma}{\sqrt{n}} = \frac{100}{\sqrt{25}} = \frac{100}{5} = 20$$

- Step 2: Calculate corresponding z-score

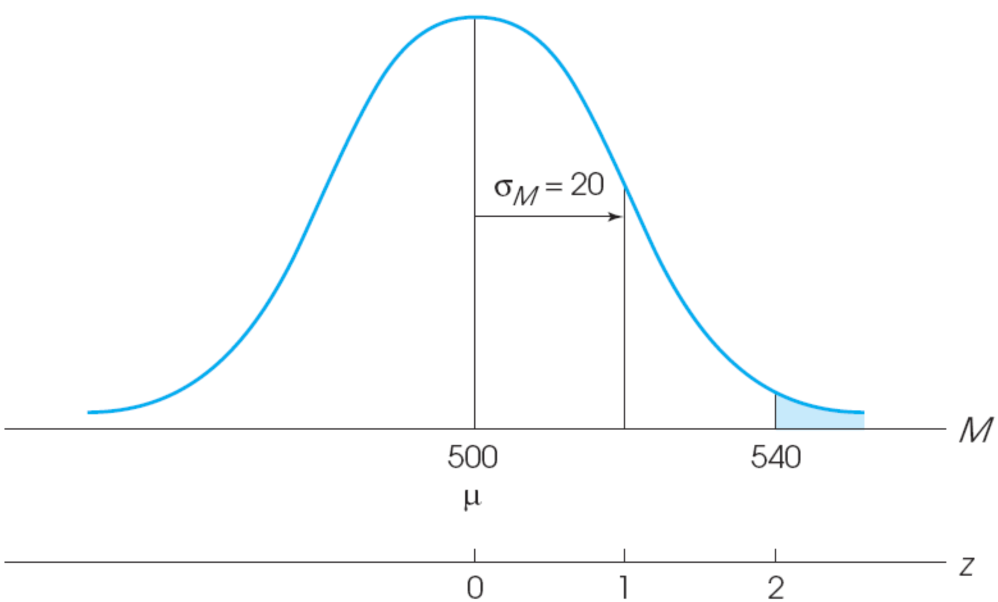
$$z = \frac{(M - \mu)}{\sigma_M} = \frac{(540 - 500)}{20} = \frac{40}{20} = 2$$

DISTRIBUTION OF SAMPLE MEANS

- Example (continued)

$$p(M > 540) = ?$$

- Step 3: Unit normal table to find correct value of p corresponding to shaded area for z



$p(M > 540) = .0228$

Z-SCORE FOR SAMPLE MEANS

- Where a sample is located relative to all other possible samples
- Formula

$$z = \frac{(M - \mu)}{\sigma_M}$$

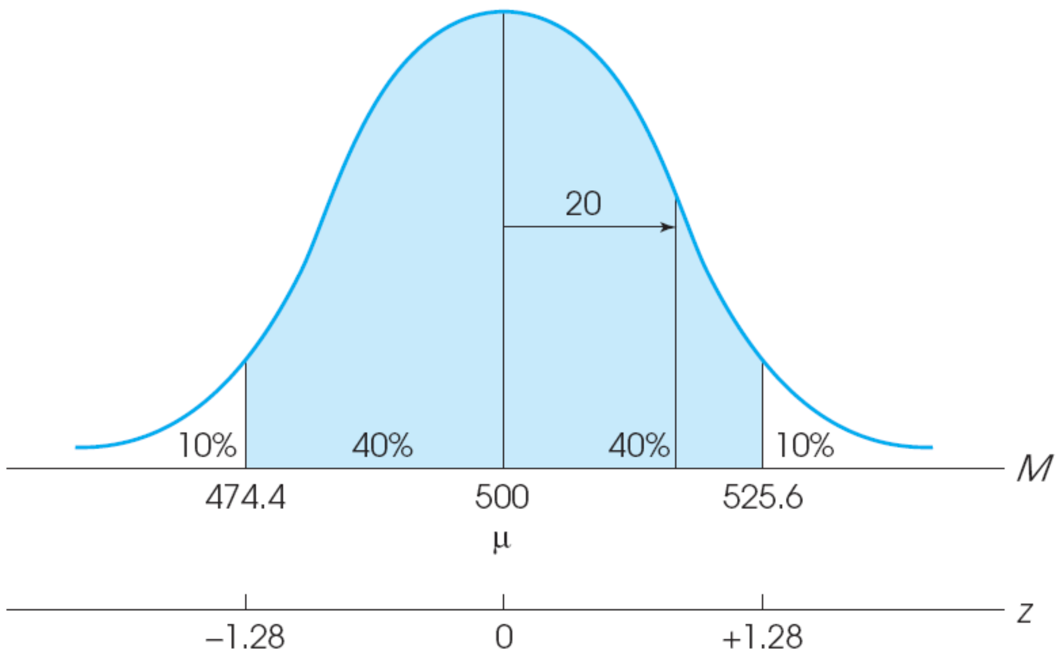
- Applications
  - Probabilities associated with specific means
  - Predict kinds of samples obtainable from a population

Z-SCORE FOR SAMPLE MEANS

- Example
  - Predict kinds of samples obtainable from a population
  - The distribution of SAT scores is normally distributed with a mean of μ = 500 and a standard deviation of σ = 100. Determine what kind of sample mean is likely to be obtained as the average SAT score for a random sample of n = 25 students 80% of the time.

Z-SCORE FOR SAMPLE MEANS

- Example (continued)
  - Determine what kind of sample mean is likely to be obtained as the average SAT score for a random sample of n = 25 students 80% of the time.



Z-SCORE FOR SAMPLE MEANS

- Example (continued)
  - Determine what kind of sample mean is likely to be obtained as the average SAT score for a random sample of n = 25 students 80% of the time.
    - z = -1.28 and 1.28
    - Last Step: Calculate mean values

$$M = \mu + z\sigma_M = 500 + (-1.28 \times 20) = 500 - 25.6 = 474.4$$

$$M = \mu + z\sigma_M = 500 + (1.28 \times 20) = 500 + 25.6 = 525.6$$

- 80% of sample means fall between **474.4** and **525.6**