Pedagogical Note: The Correlation of the Risk-Free Asset and the Market Portfolio Is Not Zero

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Abstract

The correlation coefficient between the risk-free asset and the market portfolio is undefined. However, many finance textbooks either explicitly or implicitly indicate that it is zero when discussing portfolio selection. In this paper, we discuss how the error arises and show how to correctly reach the appropriate conclusions. Hopefully, bringing attention to this issue will help instructors present the concept without confusing students.

Introduction

The correlation coefficient between the risk-free asset and the market portfolio is undefined since its calculation requires division by zero. However, when discussing the theory of portfolio selection, many finance textbooks either explicitly or implicitly indicate that the correlation coefficient between the two assets is zero. Textbooks often rely upon this incorrect assumption to show that the risk of portfolios with investment in both the risk-free asset and the market portfolio is a linear function of the market portfolio risk. Although the conclusion is valid, the correct derivation requires relying upon the covariance of the risk-free asset and market portfolio. In this paper, we discuss how the error arises and document that it occurs frequently. Our intent is not to embarrass authors, but instead to show how to present the concept to students to avoid confusion.

The Theory

Many finance textbooks begin discussion of portfolio theory with an examination of the expected return and risk of a two-asset portfolio (e.g., Ross, Westerfield, and Jaffe, 2013). The intent is to highlight the risk-return relationship when assets are combined in a portfolio. The textbooks start by showing that the expected return for any two-asset portfolio ($\bar{R}_p$) may be calculated using Equation (1) below where w_1 and w_2 are the respective percentage of total value invested in each asset and $\bar{R}_1$ and $\bar{R}_2$ are the respective expected returns for each asset. The discussion that follows emphasizes that Equation (1) shows that the expected return of any two-asset portfolio is a linear combination of the assets' expected returns.

$$\bar{R}_p = w_1\bar{R}_1 + w_2\bar{R}_2 \tag{1}$$

The discussion then directs attention to measuring portfolio risk by introducing measures of comovement of asset returns. Often covariance is introduced as a measure of the extent to which two variables move together. Its formula is shown in Equation (2) where $\sigma_{1,2}$ is the covariance of the two assets' returns, $R_{1,i}$ and $R_{2,i}$ represent the possible returns for each of the two assets, and P_i is the probability of each return occurring.

$$\sigma_{1,2} = \sum_{i=1}^{n}[R_{1,i} - \bar{R}_1][R_{2,i} - \bar{R}_2]P_i \tag{2}$$

The correlation coefficient ($\rho_{1,2}$) is also often discussed at this juncture by indicating that it is a measure of the relative comovement between returns bounded by +1.0 and -1.0. Further, the relationship between covariance and the correlation coefficient is shown as in Equations (3) and/or (4).

$$\sigma_{1,2} = \rho_{1,2}\sigma_1\sigma_2 \tag{3}$$

$$\rho_{1,2} = \frac{\sigma_{1,2}}{\sigma_1\sigma_2} \tag{4}$$

The discussion generally next indicates that the variance of a two-asset portfolio may be calculated using Equations (5) and/or (6).

$$\sigma_P^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_{1,2} \tag{5}$$

$$\sigma_P^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\rho_{1,2}\sigma_1\sigma_2 \tag{6}$$

The key observation is that Equations (5) and (6) show that the risk for the portfolio cannot be calculated as a simple linear combination of the assets' variances, but must instead also consider the assets' covariance or correlation coefficient. The equations further show that the risk of the portfolio will be reduced as the covariance or correlation coefficient becomes smaller.

Having introduced the concept using the simple two-asset portfolio, textbooks then generally turn their attention to determining optimal larger portfolios based on the approach put forth by Markowitz (1952). Markowitz showed that an investor evaluating portfolios on the basis of expected return and standard deviation (or variance) would prefer efficient portfolios that had the lowest level of risk for a given level of expected return or the largest expected return for a given level of risk (Jones, 1998). Consideration of all potential combinations of risky assets yields an efficient frontier which is the set of portfolios that offer the best risk-return combinations for investors.

The introduction of a risk-free asset extends investment options. By definition, the risk-free asset has the same return in all states of the world. Thus, the variance (and standard deviation) of the risk-free return is zero since the expected return and possible returns are the same in all states of the world. James Tobin (1958) showed that allowing lending and borrowing at the risk-free rate results in the new efficient set of portfolios being some combination of the risk-free asset and an efficient portfolio of risky-assets. The portfolio of risky-assets at the tangent point of a line emanating from the risk-free rate is the optimal risky-asset portfolio and is often referred to as the market portfolio. All investors will now hold a stake in the risk-free asset and the market portfolio with the proportion invested in each asset based on each investor's level of risk aversion. This result is the basis for the Capital Asset Pricing Model introduced by Sharpe (1963). The efficient set of portfolios of the risk-free asset and market portfolio form the Capital Market Line which depicts the highest return for each level of risk.

The important point for this paper is that all efficient portfolios consist of two assets (the risk-free asset and the market portfolio), so Equation (1) can be used to determine portfolio expected return and Equation (4) can be used to determine portfolio risk. If we use subscript *f* to denote the risk-free asset and subscript *M* to denote the market portfolio, we can restate Equations (1) and (4) below as Equation (7) and (8).

$$\bar{R}_p = w_f \bar{R}_f + w_M \bar{R}_M \tag{7}$$

$$\sigma_P^2 = w_f^2 \sigma_f^2 + w_M^2 \sigma_M^2 + 2 w_f w_M \sigma_{f,M} \tag{8}$$

As before, Equation (7) shows that the expected return for each portfolio will be a linear combination of the risk-free return and the market portfolio return with the weights being the percentage of total investment in each asset.

Note that by definition the variance of returns for the risk-free asset equals zero (σ_f^2= 0). Likewise, based on Equation (2), the covariance of the risk-free asset and the market portfolio ($\sigma_{f,M}$) must equal zero since for the risk-free asset all possible returns ($R_{f,i}$) equal the expected return for all states of the world. Thus, Equation (8) collapses to the following.

$$\sigma_P^2 = w_M^2 \sigma_M^2 \tag{9}$$

$$\sigma_P = w_M \sigma_M \tag{10}$$

Equation (10) shows that the risk of combinations of the risk-free asset and the market portfolio is a linear function determined by the fraction of total investment in the market portfolio and the risk of the market portfolio (see, Ross, Westerfield, and Jaffe 2013, p. 356; Berk and DeMarzo 2014, p. 372).

The Error

Unfortunately, many finance textbooks reach the previous conclusion in an erroneous fashion. They instead try to use Equation (6) to show the risk result. For clarity, Equation (11) restates (6) with f and M subscripts.

$$\sigma_P^2 = w_f^2\sigma_f^2 + w_M^2\sigma_M^2 + 2w_f w_M \rho_{f,M}\sigma_f\sigma_M \qquad \textbf{(11)}$$

As noted before, the covariance of the risk-free asset and the market portfolio and the standard deviation of the risk-free asset are both zero. However, using Equation (4), we see that the correlation coefficient of the risk-free asset and the market portfolio is undefined.

$$\rho_{f,M} = \frac{\sigma_{f,M}}{\sigma_f\sigma_M} = \frac{0}{0 * \sigma_M} = \frac{0}{0} \qquad \textbf{(12)}$$

This means that Equation (11) is also undefined. However, many textbook authors mistakenly assume (either explicitly or implicitly) that $\sigma_{f,M} = 0$ implies that $\rho_{f,M} = 0$.

For example, a popular text (Reilly and Brown 2012, p. 209) once used extensively in the prestigious Charted Financial Analyst program starts with a version of equation (11). The book then says since "the variance of the risk-free asset is zero" and "the correlation between the risk-free asset and any risky asset, M, is also zero" the formula can be restated in the form of Equation (9). The CFA replacement reading (Singal 2014, p.301) implicitly makes the same error by also using a version of Equation (11) to derive Equations (9) and (10). The same reading (Singal 2014, p. 342) later explicitly states that "the risk-free asset has zero correlation with the risky asset."

Note that these are not isolated cases. Many well regarded finance textbooks by highly reputable authors make the same error either explicitly by saying that the correlation coefficient between the risk-free asset and the market portfolio is zero or implicitly by using Equation (11) to obtain Equation (9). For examples, see Brigham and Daves (2014, p. 103), Brigham and Earhardt (2015, p. 992), Elton, Gruber, Brown and Goetzmann (2014, p. 82), Emery, Finnerty, and Stowe (2018, p. 158), Hearth and Zaima (2004, p. 511), Sears and Trennepohl (1993, p. 397), and Smith, Proffitt, and Stephens (1992, p. 169). Related errors may be found in journal articles. Arnold, Nail, and Nixon (2005, p. 74) recognize that the correlation of the risk-free asset with a risky asset is undefined, but mistakenly attribute the same relationship to the covariance when they state that "(b)ecause the security is risk-free, its correlation and covariance with any risky security is undefined." This list is by no means meant to be exhaustive, but it does show that the error is pervasive enough to be an important issue that instructors should address by showing that the appropriate result may be derived using Equation (8) from above.

The origin of the error is difficult to determine, but it likely results from attempts to explain the meaning of covariance and correlation. Both are measures of comovement between variables. However, covariance is an absolute measure of comovement making it more difficult to ascertain the magnitude of the relationship. The correlation coefficient indicates the relative comovement and is bounded between -1 and +1 which makes it easier to quickly recognize the degree of association. In most instances, using either measure is appropriate, but when a constant is involved, only the covariance can be calculated.

Concluding Comments

We show that many textbooks assume that the correlation of the risk-free asset with the market portfolio is zero since their covariance is zero. This incorrect assumption could confuse students since it is often used as the basis for showing that the risk of portfolios consisting of the risk-free asset and the market portfolio is a linear function. Instructors may address this error by showing students that the linear risk relationship can be derived using the covariance instead of the correlation of the two assets.

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