EVERY NON-TRIVIALLY $(2,r)$-REGULAR GRAPH IS REGULAR

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Abstract. Let $G$ be a simple graph. We say $G$ is $(2,r)$-regular if for any pair of distinct vertices $u$ and $w$, $|N(u) \cup N(w)| = r$. The graph $G$, which is neither complete nor empty, is strongly regular if it is regular and the number of vertices adjacent to two vertices $u$ and $w$ depends only on whether $u$ and $w$ are adjacent or not. In this note we prove that every non-trivially $(2,r)$-regular graph is strongly regular. As a result of this we show that if a non-trivially $(2,r)$-regular graph of order $n$ exists then $4(n-1)(n-r)+1$ is a perfect square.

1. Introduction

Let $G = (V, E)$ be a simple graph of order $n$ and let $t \leq n$. We say $G$ is a $(t,r)$-regular graph (see [3]) if for every set $S \subset V(G)$ with $|S| = t$ we have $|\bigcup_{v \in S} N(v)| = r$.

In this note we deal with non-trivially $(2,r)$-regular graphs. A $(2,r)$-regular graph is non-trivial if $r > 1$ and the graph is not complete. In [3], $(2,r)$-regular graphs were classified for $r = 0,1,2$. But characterizing $(2,r)$-regular graphs for $r \geq 3$ has proven to be a much more difficult problem (see [2, 4, 5, 6]). In [4] it is stated that the existence of non-regular $(2,r)$-regular graphs is open. In the next section we will answer this question.

Let $G = (V,E)$ be a simple graph of order $n$ which is neither complete nor empty. We say $G$ is a strongly regular $(n,d,\mu_1,\mu_2)$ graph (see [1]) if $G$ is regular ($|N(u)| = d$ for all $u \in V$) and for every distinct pair of vertices $u$ and $w$:

1. if $\{u, w\} \in E$ then $|N(u) \cap N(w)| = \mu_1$;
2. if $\{u, w\} \notin E$ then $|N(u) \cap N(w)| = \mu_2$.

Since $|N(u) \cup N(w)| = |N(u)| + |N(w)| - |N(u) \cap N(w)|$ for every distinct pair of vertices $u$ and $w$, it follows that a strongly regular $(n,d,\mu,\mu)$ graph is a $(2,2d-\mu)$-regular graph. In this note we prove that a non-trivially $(2,r)$-regular graph of order $n$ is a strongly regular $(n,d,2d-r,2d-r)$ graph for some $d$ dependent on $n$ and $r$. As an immediate result of this we show that if a non-trivially $(2,r)$-regular graph of order $n$ exists then $4(n-1)(n-r)+1$ is a perfect square.

2. $(2,r)$-regular graphs and their associated PBDs

An $(n,K,\lambda)$ pairwise balanced design (PBD) consists of an order pair $(V,B)$, where $V$ is a set of $n$ elements and $B$ is a collection of subsets of $V$ (called blocks), each of size $k \in K$, such that every distinct pair of $V$ appears in precisely $\lambda$ blocks (see [8]). The following result shows that $(2,r)$-regular graphs are related to particular PBDs. The proof of this theorem is straightforward and can be found in [6].

Theorem 1. Let $G$ be a $(2,r)$-regular graph of order $n$ with vertex set $V = \{1,2,3,\ldots,n\}$. Define $B = \{B_1,B_2,\ldots,B_n\}$ where $B_i = V \setminus N(i)$ for $i = 1, 2, \ldots, n$. Then $(V,B)$ is a PBD with $\lambda = n-r$. Moreover,

1. $i \in B_i$ for every $i \in V$;
2. $i \in B_j$ if and only if $j \in B_i$ for $i, j \in V$;
(3) \(|B_i \cap B_j| = \lambda\) for \(i, j \in V\) and \(i \neq j\).
(4) \(G\) is an \((n - k)\)-regular graph if and only if \(|B_i| = k\) for all \(i \in \{1, 2, \ldots, n\}\).

We now define an incidence matrix \(A = [a_{ij}]\), of order \(n\), for the PBD given in Theorem 1 by \(a_{ij} = 1\) if \(i \in B_j\) and \(a_{ij} = 0\) otherwise. Then:

1. \(a_{ii} = 1\) for all \(i \in V\);
2. \(A = A^t\) (that is \(a_{ij} = a_{ji}\) for all \(i, j \in V\));
3. \(A^2 = D + \lambda J\), where \(D\) is a diagonal matrix of order \(n\) with diagonal elements \(d_i = |B_i| - \lambda\) and \(J\) is the matrix of order \(n\) with every entry equal 1.

The following conjecture was stated in [6].

**Conjecture 2.** Let \(A = [a_{ij}]\) be a \((0, 1)\) symmetric matrix of order \(n\) and \(a_{ii} = 1\) for each \(i\). If the off-diagonal elements of \(A^2\) are all equal then the diagonal elements of \(A^2\) are all equal.

The authors recently discovered that Ryser in 1970 has proved ([7]) a stronger theorem. In order to state this theorem we need the following definitions. A permutation matrix is a matrix obtained by permuting the rows and columns of the identity matrix. Two matrices \(A\) and \(B\) are equivalent if there exists a permutation matrix \(P\) such that \(A = P^tBP\), where \(P^t\) is the transpose of \(P\). Define

\[
B = \begin{bmatrix}
1 & 1 & \cdots & 1 \\
1 & & & \\
\vdots & & & 0 \\
1 & & & \\
1 & & & 
\end{bmatrix} \quad (n \geq 2) \quad C = \begin{bmatrix}
0 & 1 & \cdots & 1 \\
1 & & & \\
\vdots & & & Q \\
1 & & & 
\end{bmatrix} \quad (n \geq 4)
\]

where \(0\) is the zero matrix of order \(n - 1\) and \(Q\) is a symmetric permutation matrix of order \(n - 1\). A line in a matrix is a row or a column of that matrix.

**Theorem 3.** (See [7]) Let \(A\) be a \((0, 1)\) matrix of order \(n \geq 1\) that satisfies the matrix equation \(A^2 = D + \lambda J\), where \(D\) is a diagonal matrix and \(\lambda\) is a positive integer. Then \(A\) has constant line sums \(c\) except for the \((0, 1)\) matrices \(A\) of order \(n\) with \(\lambda = 1\) equivalent to \(B\) or \(C\) and the \((0, 1)\) matrix \(A\) of order 5 with \(\lambda = 2\) equivalent to

\[
F = \begin{bmatrix}
0 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 1 \\
1 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 1 
\end{bmatrix}.
\]

Furthermore, if \(A\) has constant line sums \(c\), then \(A^2 = dI + \lambda J\), where \(c^2 = d + \lambda n\) and \(-\lambda < d \leq c - \lambda\).

**Theorem 4.** Every non-trivially \((2, r)\)-regular graph of order \(n\) is a strongly regular \((n, d, 2d - r, 2d - r)\) graph, where \(d = ((2n - 1) - \sqrt{4(n - 1)(n - r) + 1})/2\).

**Proof.** Let \(G = (V, E)\) be a non-trivially \((2, r)\)-regular graph of order \(n\) and let \((V, B)\) be the PBD given in Theorem 1. Assume \(A\) is an incidence matrix for this PBD. Then \(A\) satisfies the assumptions of Conjecture 2. If off-diagonal elements of \(A\) are all zero then \(A\) is the identity matrix. It is easy to see that \(G\) is complete in this case. This is a contradiction. Now assume that the off-diagonal entries of \(A\) are non-zero. Since \(a_{ii} = 1\), for all \(i\), it follows that \(A\) is not equivalent to matrices \(B, C\) or \(F\). Therefore \(A\) has constant line sums by
Theorem 3. (That is to say that Conjecture 2 is true.) So $|B_i| = k$ for all $i \in V$. Therefore by Part 4 of Theorem 1, $G$ is an $(n-k)$-regular graph. Thus, for every distinct pair of vertices $u$ and $w$ we have

$$|N(u) \cap N(w)| = |N(u)| + |N(w)| - |N(u) \cup N(w)| = (n-k) + (n-k) - r.$$ 

Hence $G$ is a strongly regular $(n, d, 2d - r, 2d - r)$ graph, where $d = n - k$. By Theorem 3 we also have $k^2 = (k - \lambda) + \lambda n$ or $k(k-1) = \lambda(n-1)$, where $\lambda = n - r$. Now since $k = n - d$ it follows that $(n-d)(n-d-1) = (n-r)(n-1)$ and $d = ((2n-1) - \sqrt{4(n-1)(n-r) + 1})/2$ since $d < n$. □

Corollary 5. If a non-trivially $(2, r)$-regular graph of order $n$ exists then $4(n-1)(n-r) + 1$ is a perfect square.

Example 6. The graph $K_4 \times K_4$ is a non-trivial $(2, 10)$-regular graph on 16 vertices (see for example [6]). By Theorem 4 this graph is also a strongly regular $(16, 6, 2, 2)$ graph.

We conclude this paper with the following interesting question.

Question 7. Classify all simple graphs which have the following properties. For every distinct pair of vertices $u$ and $w$:

1. if $u$ and $w$ are joined then $|N(u) \cup N(w)| = r_1$;
2. if $u$ and $w$ are not joined then $|N(u) \cup N(w)| = r_2$.

References